Artificial Intelligence 1

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Problem Solving: Informed (Heuristic) Search

Uninformed (blind) search algorithms can find an (optimal) solution to the problem, but they are usually not very efficient.

- BFS, DFS, ID, BiS

Informed (heuristic) search algorithms can find solutions more efficiently thanks to exploiting problem-specific knowledge.

- **How to use heuristics in search?**
  - BestFS, A*, IDA*, RBFS, SMA*
- **How to construct heuristics?**
  - relaxation, pattern databases

Information in search

Recall that we are looking for (the shortest) path from the initial state to some goal state.

Which information can help the search algorithm?

- For example, the length of path to some goal state.
- However such information is usually not available (if it is available then we do not need to do search). Usually some **evaluation function f(n)** is used to evaluate „quality“ of node **n** based on the length of path to the goal.
- **best-first search**
  - The node with the smallest value of f(n) is used for expansion.
- There are search algorithms with different views of **f(n)**. Usually the part of **f(n)** is a **heuristic function h(n)** estimating the length of the shortest (cheapest) path to the goal state.
  - Heuristic functions are the most common form of additional information given to search algorithms
  - We will assume that **h(n) = 0** ⇔ **n is goal**.

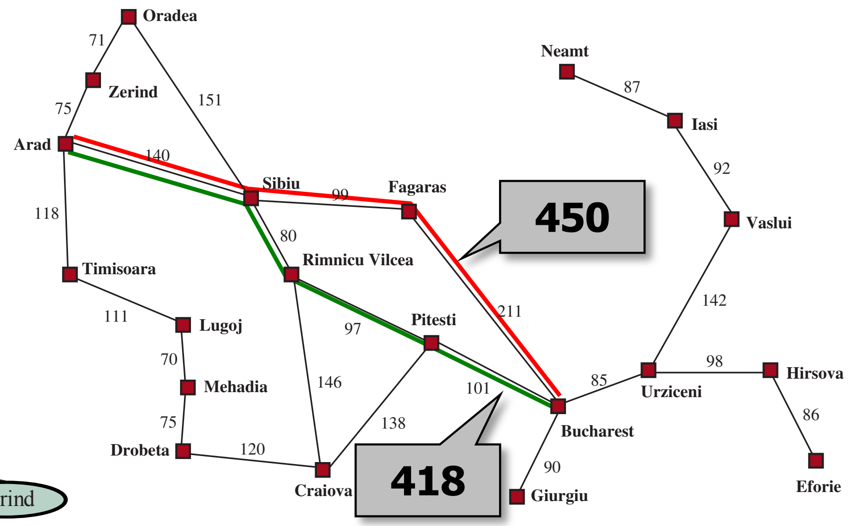
Let us try to expand first the node that is closest to some goal state, i.e. **f(n) = h(n)**.

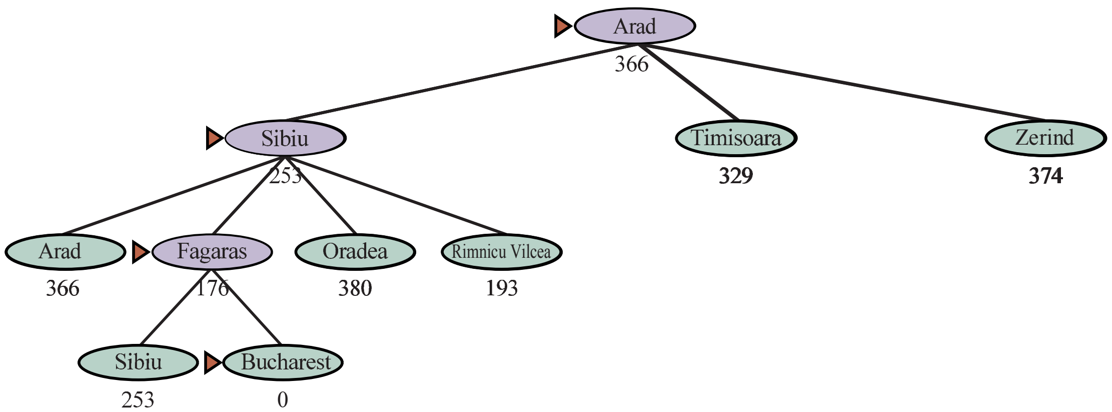
- **greedy best-first search algorithm**

Example (path Arad → Bucharest):

- We have a table of direct distances from any city to Bucharest.
- Note: this information was not part of the original problem formulation!

Arad	366	**Mehadia**	241
Bucharest	0	**Neamt**	234
Craiova	160	**Oradea**	380
Drobeta	242	**Pitesti**	100
Eforie	161	**Rimnicu Vilcea**	193
Fagaras	176	**Sibiu**	253
Giurgiu	77	**Timisoara**	329
Hirsova	151	**Urziceni**	80
Iasi	226	**Vaslui**	199
Lugoj	244	**Zerind**	374





Is it the shortest path?

We already know that the greedy algorithm **may not find the optimal path**.

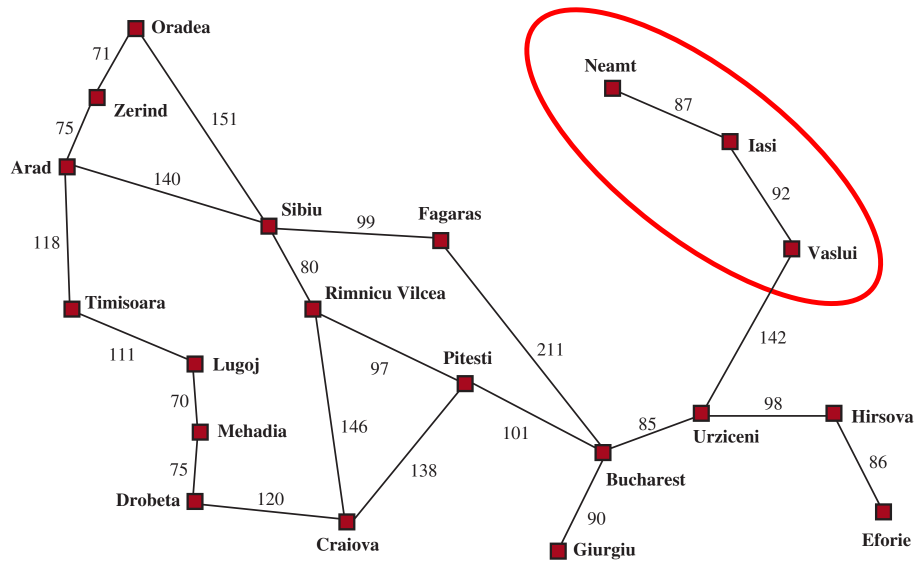
Can we at least guarantee finding some path?

- If we expand first the node with the smallest cost then the (tree search) algorithm **may not find any solution**.

  Example: path Iasi → Fagaras

  - Go to Neamt, then back to Iasi, Neamt, …
  - We need to detect repeated visits in cities!

- **Time complexity O(b^m),** where m is the maximal depth
- **Memory complexity O(b^m)**
- A good heuristic function can significantly decrease the practical complexity.



Algorithm A*

Let us now try to use **f(n) = g(n) + h(n)**

- recall that **g(n)** is the cost of path from root to **n**
- probably the most popular heuristic search algorithm
- **f(n)** represents the cost of path through **n**
- the algorithm does not extend already long paths

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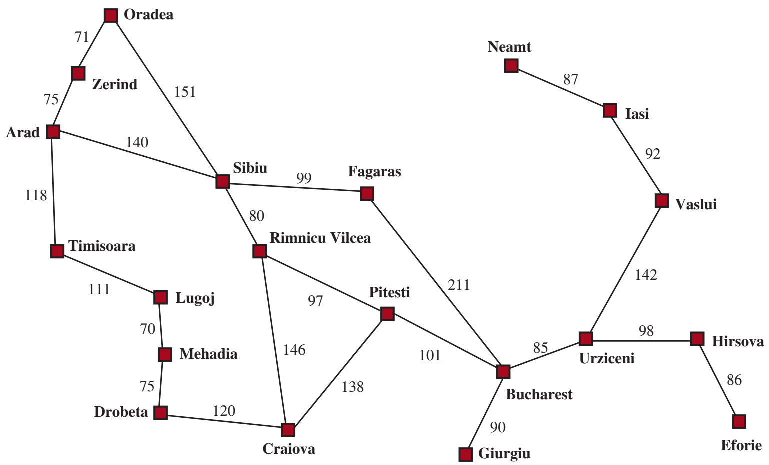
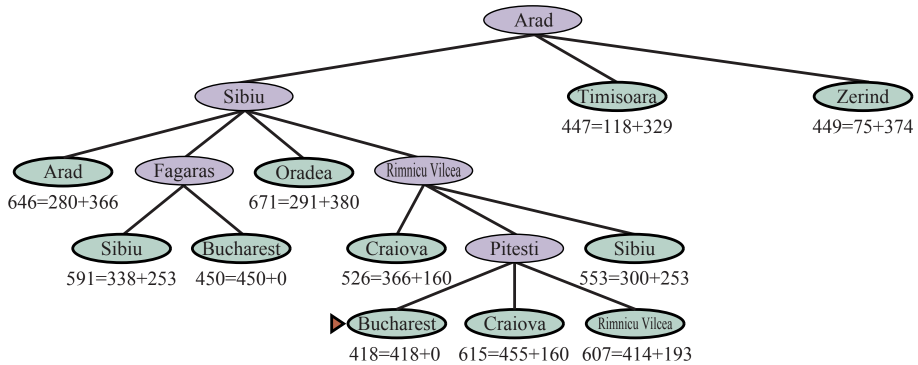
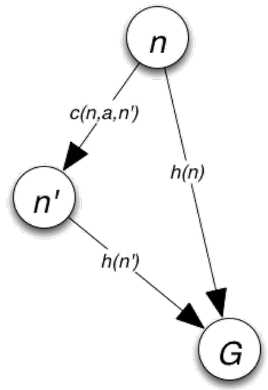




What about completeness and optimality of A*?

First a few definitions:

- **admissible heuristic h(n)**
  - **h(n) ≤ "the cost of the cheapest path from n to goal"**
  - an optimistic view (the algorithm assumes a better cost than the real cost)
  - function **f(n)** in A* is a lower estimate of the cost of path through **n**
- **monotonous (consistent) heuristic h(n)**
  - let **n'** be a successor of **n** via action **a** and **c(n,a,n')** be the transition cost
  - **h(n) ≤ c(n,a,n') + h(n')**
  - this is a form of triangle inequality



Monotonous heuristic is admissible.

let $n_1, n_2, \ldots, n_k$ be the optimal path from $\mathbf{n_1}$ to goal $\mathbf{n_k}$, then

$h(n_i) - h(n_{i+1}) \leq c(n_i,a_i,n_{i+1})$, via monotony

$h(n_1) \leq \Sigma_{i=1,..,k-1}\, c(n_i,a_i,n_{i+1})$, after „sum"

For a monotonous heuristic the values of f(n) are non-decreasing over any path.

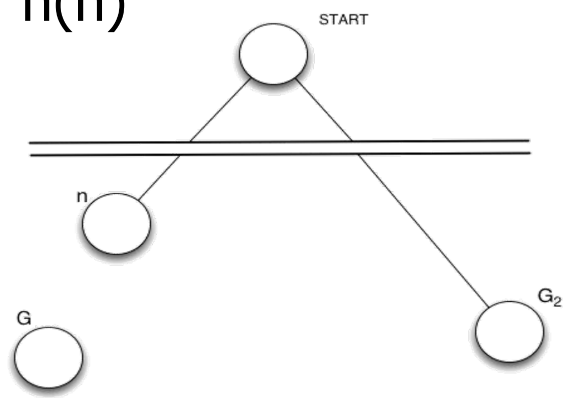
Let **n'** be a successor of **n**, i.e. $g(n') = g(n) + c(n,a,n')$, then

$\mathbf{f(n')} = g(n') + h(n') = g(n) + c(n,a,n') + h(n') \geq g(n) + h(n) = \mathbf{f(n)}$

If h(n) is an admissible heuristic then the algorithm A* in TREE-SEARCH is optimal.

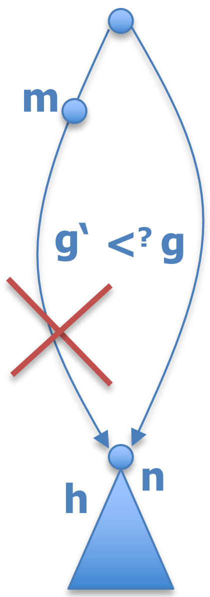
- in other words – the first expanded goal is optimal
- Let G_2 be sub-optimal goal from the fringe and C* be the optimal cost
  - $f(G_2) = g(G_2) + h(G_2) = g(G_2) > C^*$, because $h(G_2) = 0$
- Let **n** be a node from the fringe and being on the optimal path
  - $f(n) = g(n) + h(n) \leq C^*$, via admissibility of h(n)
- together
  - $f(n) \leq C^* < f(G_2)$,

  i.e., the algorithm must expand **n** before G_2 and this way it finds the optimal path.



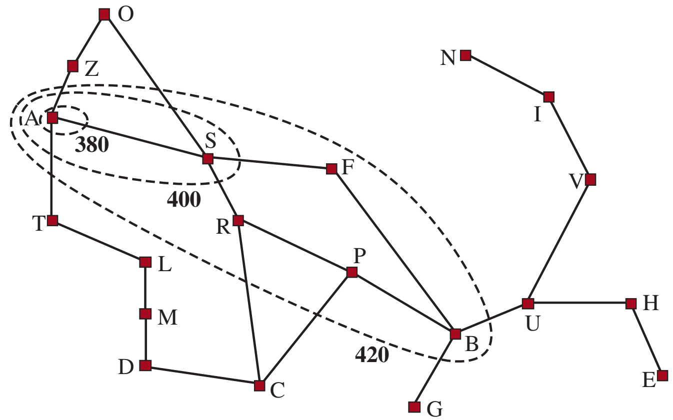
If h(n) is a monotonous heuristic then the algorithm A* in GRAPH-SEARCH is optimal.

- Possible **problem**: reaching the same state for the second time using a better path – classical GRAPH-SEARCH ignores this second path!
- Possible **solution**: selection of the better of the two paths leading to the closed node (extra bookkeeping) or using monotonous heuristic.
  - for monotonous heuristics, the values of f(n) are non-decreasing over any path
  - A* selects for expansion the node with the smallest value of f(n), i.e., the values f(m) of other open nodes **m** are not smaller, i.e., among all "open" paths to **n** there cannot be a shorter path than the path just selected (no path can shorten)
  - hence, the first closed goal node is optimal



For non-decreasing function f(n) we can draw **contours** in the state graph (the nodes inside a given contour have f-costs less than or equal to the contour value.

- for h(n) = 0 we obtain circles around the start
- for more accurate h(n) we use, the bands will stretch toward the goal state and become more narrowly focused around the optimal path.



- A* expands all nodes such that **f(n) < C*** on the contour
- A* can expand some nodes such that **f(n) = C***
- the nodes **n** such that **f(n) > C*** are never expanded
- the algorithm A* is **optimally efficient** for any given consistent heuristic

Time complexity:

A* can expand an exponential number of nodes

- this can be avoided if |h(n)-h*(n)| ≤ O(log h*(n)), where h*(n) is the cost of optimal path from n to goal

Space complexity:

A* keeps in memory all expanded nodes

A* usually runs out of space long before it runs out of time

A simple way to decrease memory consumption is iterative deepening.

Algorithm IDA*

```
function IDA*(problem) returns a solution sequence
  inputs: problem, a problem
  static: f-limit, the current f- COST limit
          root, a node

  root ← MAKE-NODE(INITIAL-STATE[problem])
  f-limit ← f- COST(root)
  loop do
      solution, f-limit ← DFS-CONTOUR(root, f-limit)
      if solution is non-null then return solution
      if f-limit = ∞ then return failure; end
--------------------------------------------------------------------
function DFS-CONTOUR(node, f-limit) returns a solution sequence and a new f- COST limit
  inputs: node, a node
          f-limit, the current f- COST limit
  static: next-f, the f- COST limit for the next contour, initially ∞

  if f- COST[node] > f-limit then return null, f- COST[node]
  if GOAL-TEST[problem](STATE[node]) then return node, f-limit
  for each node s in SUCCESSORS(node) do
      solution, new-f ← DFS-CONTOUR(s, f-limit)
      if solution is non-null then return solution, f-limit
      next-f ← MIN(next-f, new-f); end
  return null, next-f
```

- the search limit is defined using the cost **f(n)** instead of depth
- for the next iteration we use the smallest value **f(n)** of node **n** that exceeded the limit in the last iteration
- frequently used algorithm

Recursive best-first search

Let us try to mimic standard best-first search, but using only linear space

- the algorithm stops exploration if there is an alternative path with better cost f(n)
- when the algorithm goes back to node **n**, it replaces the value f(n) using the cost of successors (remembers the best leaf in the forgotten subtree)

If h(n) is an admissible heuristic then the algorithm is optimal.

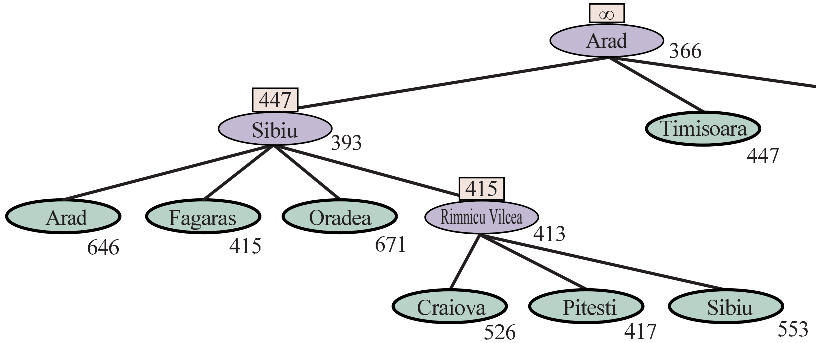
- **Space complexity O(bd)**
- **Time complexity is still exponential** (suffers from excessive node re-generation)

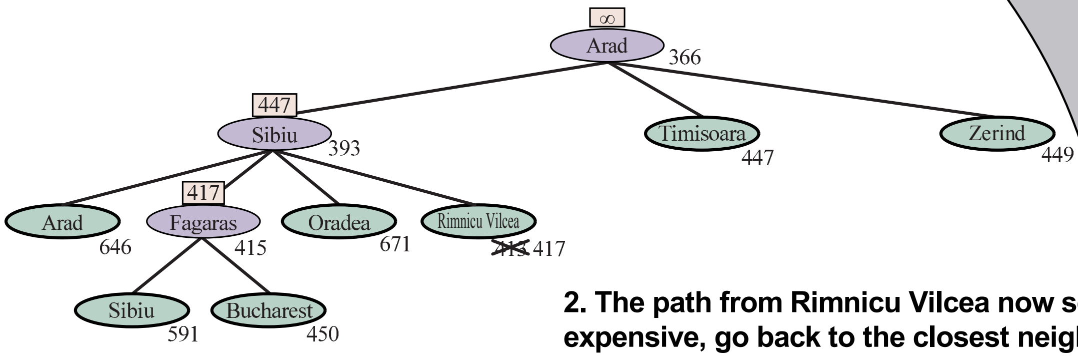
```
function RECURSIVE-BEST-FIRST-SEARCH(problem) returns a solution, or failure
    RBFS(problem, MAKE-NODE(INITIAL-STATE[problem]), ∞)

function RBFS(problem, node, f_limit) returns a solution, or failure and a new f-cost limit
    if GOAL-TEST[problem](STATE[node]) then return node
    successors ← EXPAND(node, problem)
    if successors is empty then return failure, ∞
    for each s in successors do
        f[s] ← max(g(s) + h(s), f[node])
    repeat
        best ← the lowest f-value node in successors
        if f[best] > f_limit then return failure, f[best]
        alternative ← the second-lowest f-value among successors
        result, f[best] ← RBFS(problem, best, min(f_limit, alternative))
        if result ≠ failure then return result
```

Recursive BFS: example

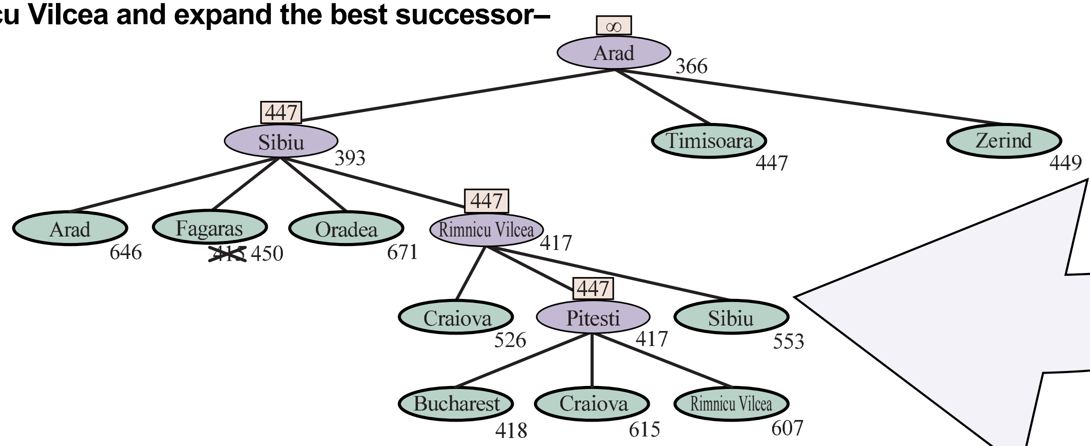
1. After expansion of Arad, Sibia, Rimnicu Vilcea





2. The path from Rimnicu Vilcea now seems too expensive, go back to the closest neighbour – Fagaras
a more accurate cost is stored for Rimnicu Vilcea

3. The path through Fagaras is now worse, go back to Rimnicu Vilcea and expand the best successor– Pitesti



IDA* and RBFS do not exploit available memory!

This is a pity as the already expanded nodes are re-expanded again (waste of time)

Let us try to modify classical A*

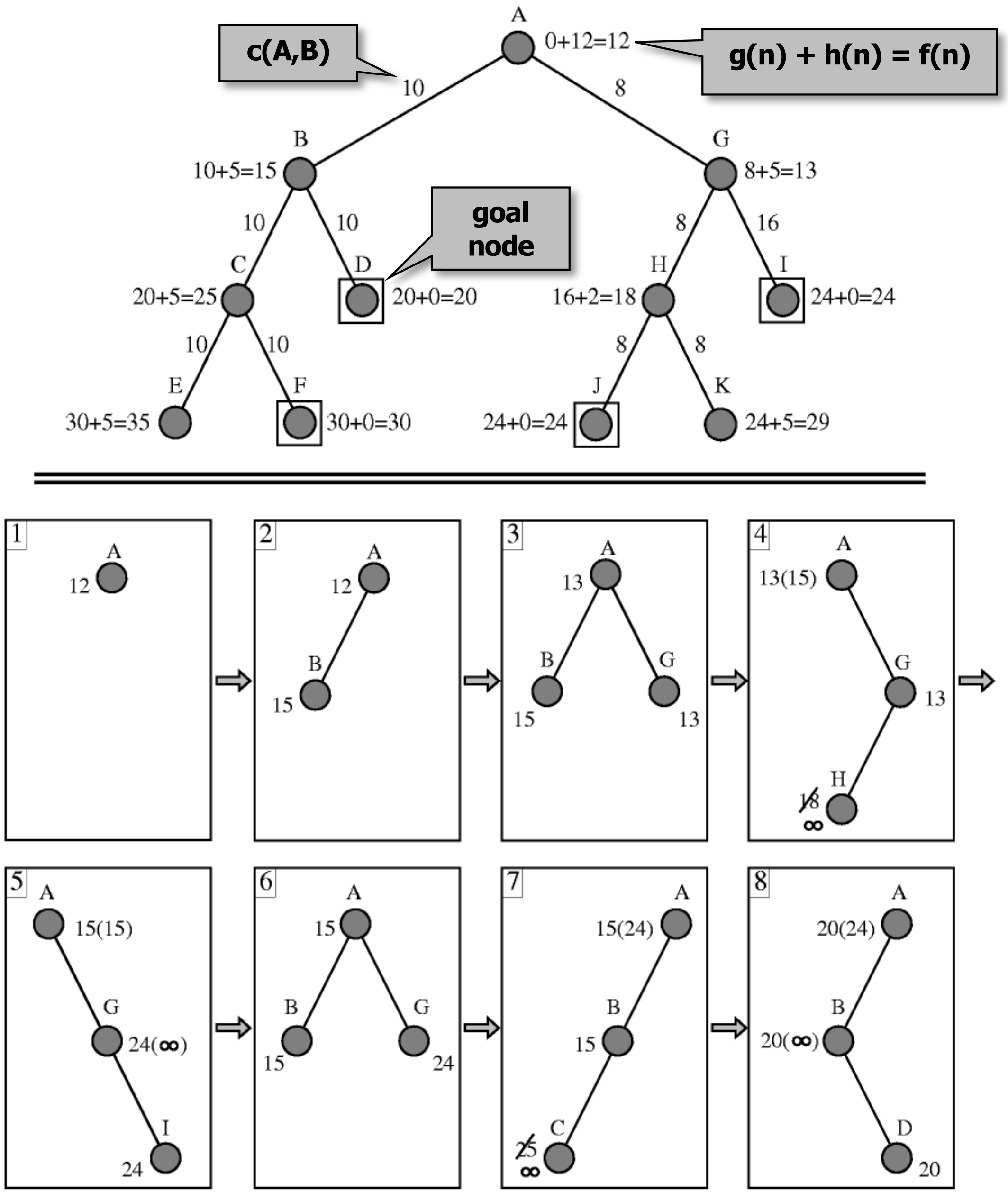
```
function SMA*(problem) returns a solution sequence
  inputs: problem, a problem
  static: Queue, a queue of nodes ordered by f-cost

  Queue ← MAKE-QUEUE({MAKE-NODE(INITIAL-STATE[problem])})
  loop do
     if Queue is empty then return failure
     n ← deepest least-f-cost node in Queue
     if GOAL-TEST(n) then return success
     s ← NEXT-SUCCESSOR(n)
     if s is not a goal and is at maximum depth then
        f(s) ← ∞
     else
        f(s) ← MAX(f(n), g(s)+h(s))
     if all of n's successors have been generated then
        update n's f-cost and those of its ancestors if necessary
     if SUCCESSORS(n) all in memory then remove n from Queue
     if memory is full then
        delete shallowest, highest-f-cost node in Queue
        remove it from its parent's successor list
        insert its parent on Queue if necessary
     insert s on Queue
  end
```

Path from root to this non-goal node can be stored in memory, hence no optimal path through this node can be found.

- when memory is full, drop the worst leaf node – the node with the highest f-value (if there are more such nodes then drop the shallowest node)
- similarly to RBFS back up the value of the forgotten node to its parent

Simplified memory-bounded A*: example



- Assume memory for **three nodes** only.
- If there is enough memory to store an optimal path then SMA* finds an optimal solution.
- Otherwise it finds the best path with available memory.
  - If the cost of J would be 19, then this is optimal goal, but the path to it can not be stored in memory!

Weighted A* (satisficing search)

A* still expands a lot of nodes (to guarantee optimality).

If we are willing to accept suboptimal solutions (good enough or **satisficing solutions**), we can explore fewer nodes.

How? We allow inadmissible heuristics.

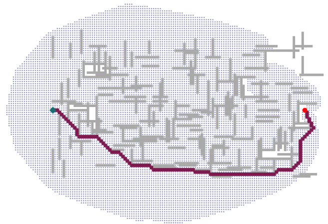
Weighted A*

f(n) = g(n) + W x **h(n),** for some W > 1

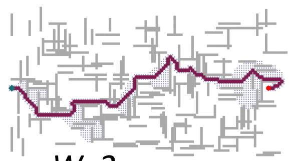
Finds solutions with the cost between C* and W x C* (in practice, the cost is closer to C* than to W x C*).

Algorithm	f(n)	W
A* search	g(n) + h(n)	W = 1
Uniform-cost search	g(n)	W = 0
Greedy best-first search	h(n)	W = ∞
Weighted A* search	g(n) + W x h(n)	1 < W < ∞

A*

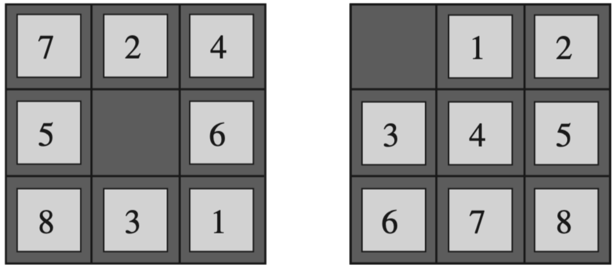


Weighted A*



W=2
7 times fewer states
5% mostly costly path

How to find admissible heuristics?



Start State   Goal State

Example: 8-puzzle

- 22 steps to goal in average
- branching factor around 3
- (complete) search tree: $3^{22} \approx 3{,}1 \times 10^{10}$ nodes
- the number of reachable states is only 9!/2 = 181 440
- for 15-puzzle there are 10^{13} states
- we need some heuristic, preferable admissible
  - h_1 = „the number of misplaced tiles“
    = 8
  - h_2 = „the sum of the distances of the tiles from the goal positions“
    = 3 + 1 + 2 + 2 + 2 + 3 + 3 + 2 = 18

    a so called Manhattan heuristic
  - the optimal solution needs 26 steps

How to characterize the quality of a heuristic?

Effective branching factor b*

- Let the algorithm need **N** nodes to find a solution in depth **d**
- **b*** is a branching factor of a uniform tree of depth **d** containing **N+1** nodes

$$N+1 = 1 + b^* + (b^*)^2 + \dots + (b^*)^d$$

Example:

- 8-puzzle
- the average over 100 instances for each of various solution lengths

	Search Cost (nodes generated)			Effective Branching Factor		
d	BFS	$A^*(h_1)$	$A^*(h_2)$	BFS	$A^*(h_1)$	$A^*(h_2)$
6	128	24	19	2.01	1.42	1.34
8	368	48	31	1.91	1.40	1.30
10	1033	116	48	1.85	1.43	1.27
12	2672	279	84	1.80	1.45	1.28
14	6783	678	174	1.77	1.47	1.31
16	17270	1683	364	1.74	1.48	1.32
18	41558	4102	751	1.72	1.49	1.34
20	91493	9905	1318	1.69	1.50	1.34
22	175921	22955	2548	1.66	1.50	1.34
24	290082	53039	5733	1.62	1.50	1.36
26	395355	110372	10080	1.58	1.50	1.35
28	463234	202565	22055	1.53	1.49	1.36

Is h_2 (from 8-puzzle) always better than h_1 and how to recognize it?

- notice that $\forall n \; h_2(n) \geq h_1(n)$
- we say that **h_2 dominates h_1**
- A* with h_2 never expands more nodes than A* with h_1
  - A* expands all nodes such that $f(n) < C^*$, so $h(n) < C^* - g(n)$
  - In particular if it expands a node using h_2, then the same node must be expanded using h_1

It is always better to use a heuristic function giving higher values provided that

- **the limit $C^* - g(n)$ is not exceeded** (then the heuristic would not be admissible)
- **the computation time is not too long**

Can an agent construct admissible heuristics for any problem?

Yes, via **problem relaxation**!

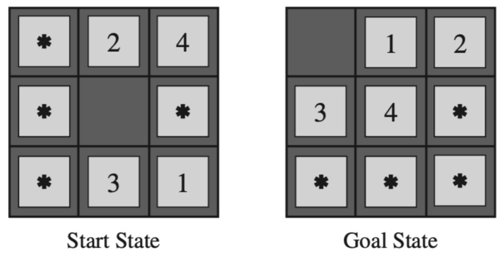
- relaxation is a simplification of the problem such that the solution of the original problem is also a solution of the relaxed problem (even if not necessarily optimal)
- we need to be able to solve the relaxed problem fast
- the cost of optimal solution to a relaxed problem is a lower bound for the solution to the original problem and hence it is an admissible (and monotonous) heuristic for the original problem

Example (8-puzzle)

- a tile can move from square A to square B if:
  - A is horizontally or vertically adjacent to B
  - B is blank
- possible relaxations (omitting some constraints to move a tile):
  - a tile can move from square A to square B if A is adjacent to B (Manhattan distance)
  - a tile can move from square A to square B if B is blank
  - a tile can move from square A to square B (heuristic h_1)

Another approach to admissible heuristics is using a **pattern database**

- based on solution of specific sub-problems (patterns)
- by searching back from the goal and recording the cost of each new **pattern** encountered



- heuristic is defined by taking the worst cost of a pattern that matches the current state
- Beware! The "sum" of costs of matching patterns needs not be admissible (the steps for solving one pattern may be used when solving another pattern).

If there are **more heuristics**, we can always use the **maximum** value from them (such a heuristic dominates each of the used heuristics).

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