Complexity of permanents

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I. Conjecture: Permanent is exponentially harder than determinant.

$Per(A) = \sum_{\pi} \prod_{i} A_{i,\pi(i)}$.

If G is bipartite graph then the permanent of its adjacency matrix is equal to the number of perfect matchings of G.

We consider matrix $A = (A_{i,j})$ as matrix of variables; $\det(A), Per(A)$ are thus multivariable polynomials with each coefficient 1 or -1.

- Formula size of a polynomial: minimum number of additions and multiplications needed to get the polynomial startring from the variables.
- Valiant: Determinant complexity of a polynomial: min size of a matrix A so that the polynomial equals $\det(A)$ after substitution of some $A_{i,j}$'s by other variables or real constants.
- Theorem (Valiant). Determinant complexity is at most twice formula size.

Does permanent have exponential determinant complexity?

Theorem

(Kasteleyn 61; Galluccio, Loebl 89; Tessler 90; Cimasoni, Reshetikhin 2002)

$$Per(A) = 2^{-g} \sum_{i=1}^{4^g} s_i \det(A_i),$$

where $s_i \in \{1, -1\}$ and each A_i is obtained from A by change of sign of some entries. Here g is genus of the bipartite graph whose adjacency matrix is A.

Aditive determinant complexity: What is minimum number of signings A_i so that $Per(A)$ is linear combination of their determinants?

Norine made conjecture in 2004 that the answer is always power of 4 (4^g) but it was disproved by Miranda and Lucchesi.

It is not known whether aditive determinant complexity of the permanent is exponential in the size of the matrix.

A set E' of edges of graph $G = (V, E)$ is *even* if graph (V, E') has all degrees even. For example, the empty set is even.

Graph $G = (V, E)$ variable x_e associated with each edge e, $x = (x_e)_{e \in E}$.

Ising partition function is

$$\mathbf{E}(G, x) = \sum_{E' \subset E \text{ even}} \prod_{e \in E'} x_e.$$

There is a natural way to define basic sign $s(E')$ for each even set of edges; we let

$$\mathbf{E}^s(G, x) = \sum_{E' \subset E \text{ even}} s(E') \prod_{e \in E'} x_e.$$

Aditive determinant complexity of Ising partition function

Aditive determinant complexity of $\mathbf{E}(G, x)$: minimum number c of sets of edges $S_i, i = 1, \ldots, c$ of G so that $\mathbf{E}(G, x)$ is linear combination of

$$\mathbf{E}^i(G, x) = \sum_{E' \subset E \text{ even}} s(E')(-1)^{|E' \cap S_i|} \prod_{e \in E'} x_e.$$

Theorem

(Loebl, Masbaum, 2011) Aditive determinat complexity of Ising partition function is 4^g.

Is there a relation of determinant and aditive determinat complexity? Joint work with Gregor Masbaum.

- Preliminary result: For each graph G there is a graph H (exponentially bigger) so that $\mathbf{E}(G, x)$ is obtained from $\mathbf{E}^s(H, x)$ by substitutions of real constants or $x_e, -x_e$ for variables of edges of H.
- Mixed model: assume graph H has k vertex-disjoint subgraphs izomorphic to graph G. We let variables to be the same in each copy of G, and we let them to be 1 outside of graphs G.

  Question. What is the minimum number of signings of $\mathbf{E}^s(H, x)$ so that $\mathbf{E}(G, x)$ is their linear combination?

It seems that it is more natural to study this question in the setting of *discrete Ihara-Selberg function*

$$I(G,M)=\prod_p(1-\prod_{t \text{ transition of } p} M(t)),$$

where the (infinite) product ranges over *aperiodic reduced closed walks* p on G and M is matrix *of transitions between orientations of edges* of G.

- Feynman noticed and Sherman proved in the beginning of 60' that $\mathbf{E}(G,x)^2$ for planar graph G is equal to $I(G,M)$ where transition matrix M is determined by the rotation.
- Bass proved in 80's that $I(G,M)$ is a determinant.

The permanent of a $n \times n \times n$ 3D matrix A is defined to be

$$Per(A) = \sum_{\sigma_1, \sigma_2 \in S_n} \prod_{i=1}^{n} a_{i\sigma_1(i)\sigma_2(i)}.$$

The determinant of a $n \times n \times n$ 3D matrix A is defined to be

$$\det(A) = \sum_{\sigma_1, \sigma_2 \in S_n} sign(\sigma_1) sign(\sigma_2) \prod_{i=1}^{n} a_{i\sigma_1(i)\sigma_2(i)}.$$

Each Latin square corresponds to a 3-partite hypergraph where each entry produces one triple (row, column, number).

Hence Ryser, Brualdi, Stein conjecture is equivalent to assertion that $Per(A) \neq 0$ for the correponding 3D-matrix A.

Kasteleyn 3D-matrices

Rich class of 3D-matrices have aditive determinat complexity 1.

We say that an $n \times n \times n$ 3D matrix A is Kasteleyn if there is 3D matrix A' obtained from A by changing signs of some entries so that $Per(A) = \det(A')$.

Theorem

(Loebl, Rytir 2012) Let M be $n \times n$ matrix. Then one can construct $m \times m \times m$ Kasteleyn 3D matrix A with $m \le n^2 + 2n$ and $Per(M) = Per(A)$. Moreover, Kasteleyn signing is trivial, i.e., $Per(A) = \det(A)$, and if M is non-negative then A is non-negative.

Binet- Cauchy formula.

Lemma

(follows simply from considerations of Barvinok) Let A^1, A^2, A^3 be real $r \times n$ matrices, $r \leq n$. For a subset $I \subset \{1, \ldots, n\}$ of cardinality r we denote by A^s_I the $r \times r$ submatrix of the matrix A^s consisting of the columns of A^s indexed by the elements of the set I. Let C be the 3D matrix defined, for all i_1, i_2, i_3 by

$$C_{i_1,i_2,i_3} = \sum_{j=1}^{n} A^1_{i_1,j} A^2_{i_2,j} A^3_{i_3,j}.$$

Then

$$\det(C) = r! \sum_I Per(A^1_I) \det(A^2_I) \det(A^3_I),$$

where the sum is over all subsets $I \subset \{1, \ldots, n\}$ of cardinality r.