Linear extension diameter of level induced subposets of the Boolean lattice

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Abstract

The linear extension diameter of a finite poset $\mathcal{P}$ is the diameter of the graph on all linear extensions of $\mathcal{P}$ as vertices, two of them being adjacent whenever they differ in a single adjacent transposition. We determine the linear extension diameter of the subposet of the Boolean lattice induced by the 1st and kth levels and describe an explicit construction of all diametral pairs. This partially solves a question of Felsner and Massow. The diametral pairs are obtained from minimal vertex-edge covers of so called dependency graphs, a new concept which may be of independent interest.

1 Introduction

The linear extension graph $G(\mathcal{P})$ of a finite poset $\mathcal{P}$ has all its linear extensions as vertices, two of them being adjacent whenever they differ in a single adjacent transposition. Linear extension graphs were first defined by Pruesse and Ruskey [7] who considered the problem of generating all linear extensions of a poset $\mathcal{P}$ by (adjacent) transpositions; that is, finding Hamiltonian path in $G(\mathcal{P})$. An explicit study of structural properties of linear extension graphs was started by Björner and Wachs [1] and by Reuter [8]; see also [6]. Among its properties, let us mention that the linear extension graph of any poset is a partial cube; that is, an isometric subgraph of a hypercube. Incomparable pairs of the poset correspond to directions in the minimal hypercube into which the linear extension graph isometrically embeds.

The linear extension diameter of a finite poset $\mathcal{P}$, denoted by $\mathrm{led}(\mathcal{P})$, is the diameter of $G(\mathcal{P})$. It equals the maximum number of pairs of $\mathcal{P}$ that appear in a reversed order in two linear extensions of $\mathcal{P}$. In other words, it is the maximum number of incomparable pairs in a 2-dimensional extension of $\mathcal{P}$. The linear extension diameter was introduced by Felsner and Reuter [4] who investigated its relation to other poset parameters such as height, width, fractional dimension and other properties. They also conjectured that the linear extension diameter of the Boolean lattice is

$$\mathrm{led}(\mathcal{B}_n) = 2^{2n-2} - (n+1)2^{n-2}. \tag{1}$$

Felsner and Massow [3] proved this conjecture by an (elegant) combinatorial argument and characterized all diametral pairs of linear extensions of $\mathcal{B}_n$. They are formed by a reversed

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lexicographical order with respect to some permutation σ of atoms (shortly σ-revlex) and a $\overline{\sigma}$-revlex order where $\overline{\sigma}$ denotes the reverse of σ. Moreover, they extended this characterization to a more general class of downset lattices of 2-dimensional posets.

Brightwell and Massow [2] show that determining the linear extension diameter of a given poset is NP-complete problem. Interestingly, diametral pairs can be used to obtain optimal drawings of the poset [3]. For further properties of linear extension graphs and the linear extension diameter the interested reader is referred to a dissertation of Massow [5] and the references within.

In this paper we determine the linear extension diameter of the subposet $\mathcal{B}_n^{1,k}$ of the Boolean lattice $\mathcal{B}_n$ induced by the 1st and kth levels and we describe an explicit construction of all diametral pairs of linear extensions. This partially solves a question of Felsner and Massow [3] on diametral pairs of subposets of the Boolean lattice induced by two levels.

Theorem 1. *For every* $1 < k \leq n$*,*

$$\operatorname{led}(\mathcal{B}_n^{1,k}) = \binom{\binom{n}{k}}{2} + 2\binom{n}{k+1} + \binom{n}{2} - \sum_{\substack{i=k \\ i \equiv n \pmod 2}}^{n-2} \binom{i}{k}.$$

Almost all diametral pairs are formed by two linear extensions that reverse all pairs of atoms, all pairs of k-sets and certain pairs of an atom and a k-set that correspond to a minimal vertex cover of so called *dependency graph*, which is defined below. For a precise characterization of diametral pairs see Theorems 17 and 19. Our approach in fact allows to determine the maximal distance between two linear extensions with fixed orders of atoms in terms of the minimal size of a vertex cover of the respective dependency graph. The concept of dependency graphs is new, as far as we know, and may be of independent interest.

2 Preliminaries

The distance $d(L_1, L_2)$ between two linear extensions L_1, L_2 in the linear extension graph $G(\mathcal{P})$ of a finite poset $\mathcal{P}$ equals the number of pairs of elements of $\mathcal{P}$ that appear in L_1 and L_2 in a reversed order. Such pair is called a *reversal* (or a *reversed pair*). Clearly, a reversal can be only an incomparable pair of $\mathcal{P}$.

The poset $\mathcal{B}_n^{1,k}$ where $1 < k \leq n$ consists of all singletons (called atoms) and k-sets over $[n] = \{1, \ldots, n\}$, ordered by inclusion. Every automorphism of $\mathcal{B}_n^{1,k}$ is obtained as a (unique) extension of some permutation of atoms. Note that in general, an automorphism of $\mathcal{P}$ leads to an automorphism of $G(\mathcal{P})$. We use letters $S, T, \ldots$ to denote subsets of $[n]$ whereas $u, v, \ldots$ denotes the elements from $[n]$. For ease of notation, let us write k-sets of $[n]$ compactly; for example $\{u, v, w\}$ as uvw. (Thus, uv represents the 2-set $\{u, v\}$ whereas $\{u, v\}$ represents the pair $\{\{u\}, \{v\}\}$ of atoms.)

For a permutation σ of $[n]$ we write $u <_\sigma v$ if u is before v in σ; that is, $\sigma^{-1}(u) < \sigma^{-1}(v)$. For a set $S \subseteq [n]$ let $\max_\sigma(S)$ denote the maximum in S with respect to $<_\sigma$. Analogously we define $\min_\sigma(S)$. Furthermore, let $\overline{\sigma}$ denote the reverse of σ and let $\operatorname{inv}(\sigma)$ be the number of inversions in σ. For example, for $\sigma = 2341$ we have $3 <_\sigma 1$, $\overline{\sigma} = 1432$, and $\operatorname{inv}(\sigma) = 3$.

Let $\mathcal{L}_n^k(\sigma)$ for $1 < k \leq n$ and a permutation σ of $[n]$ be the set of all linear extensions of $\mathcal{B}_n^{1,k}$ with the order of atoms preserving the relation $<_\sigma$. Note that the indices k and n may be omitted whenever they are clear from the context. When looking for a diametral pair of linear extensions L_1, L_2 in $G(\mathcal{B}_n^{1,k})$, we may assume without loss of generality that $L_1 \in \mathcal{L}(id)$ where id denotes the identity permutation. All other diametral pairs can be obtained by automorphisms of $\mathcal{B}_n^{1,k}$.

Let σ be a fixed permutation of $[n]$, $1 < k \leq n$, $L_1 \in \mathcal{L}_n^k(id)$, and $L_2 \in \mathcal{L}_n^k(\sigma)$. The order in L_1 and L_2 is denoted by $<_{L_1}$ and $<_{L_2}$, respectively. Clearly, the number of pairs of atoms reversed in L_1, L_2 is $\mathrm{inv}(\sigma)$, the number of inversions in σ. For a k-set S and an atom $u \notin S$, the pair $\{S, u\}$ is said to be

- *free* in $\mathcal{L}(\sigma)$ if $u >_\sigma v$ for every $v \in S$; otherwise, it is *fixed* in $\mathcal{L}(\sigma)$,
- *reversible* if it is free in $\mathcal{L}(\sigma)$ or in $\mathcal{L}(id)$; otherwise it is *nonreversible*.
- *simple* if it is free both in $\mathcal{L}(\sigma)$ and in $\mathcal{L}(id)$; otherwise it is *nonsimple*.

For example, for $\sigma = 2341$ and $k = 2$ the pair $\{23, 1\}$ is free in $\mathcal{L}(\sigma)$ and fixed in $\mathcal{L}(id)$; hence it is reversible and nonsimple. The pair $\{34, 2\}$ is fixed both in $\mathcal{L}(\sigma)$ and $\mathcal{L}(id)$; hence it is nonreversible. The pair $\{23, 4\}$ is free both in $\mathcal{L}(\sigma)$ and $\mathcal{L}(id)$; hence it is reversible and simple.

If $\{S, u\}$ is free in $\mathcal{L}(\sigma)$, there exists $L \in \mathcal{L}(\sigma)$ such that $S <_L u$; otherwise, $u <_L S$ for every $L \in \mathcal{L}(\sigma)$. If $\{S, u\}$ is not reversible, then $\{S, u\}$ cannot be reversed in any pair of linear extensions $L_1 \in \mathcal{L}(id)$, $L_2 \in \mathcal{L}(\sigma)$. Clearly $\{S, u\}$ is simple if and only if $u >_{id} v$ and $u >_\sigma v$ for every $v \in S$. Let $s(\sigma)$ denote the number of simple pairs with respect to σ. Clearly, the number of all reversible pairs is

$$2\binom{n}{k+1} - s(\sigma).$$

Indeed, every $(k+1)$-set T corresponds to reversible pairs $\{T \setminus \{\max_{id}(T)\}, \max_{id}(T)\}$ and $\{T \setminus \{\max_\sigma(T)\}, \max_\sigma(T)\}$, which are distinct if and only if they are nonsimple.

Ideally, for a given permutation σ we would like to find a pair of linear extensions $L_1 \in \mathcal{L}(id)$, $L_2 \in \mathcal{L}(\sigma)$ such that every pair of k-sets and every reversible pair (of an atom and a k-set) is reversed in L_1, L_2; that is, we would like $d(L_1, L_2) = \mathrm{inc}(\sigma)$ where

$$\mathrm{inc}(\sigma) = \binom{\binom{n}{k}}{2} + 2\binom{n}{k+1} + \mathrm{inv}(\sigma) - s(\sigma)$$

is the number of incomparable pairs in the intersection poset of all linear extensions from $\mathcal{L}(id)$ or $\mathcal{L}(\sigma)$. However, we will see in the next section that by Proposition 3, this is possible only if the dependency graph $G(\sigma)$ has no edge; that is, $k = n$ or, $k = n - 1$ and $\sigma(n) \neq n$.

In general, we study how close to $\mathrm{inc}(\sigma)$ we can get. Therefore, we define the following number of not reversed pairs in L_1, L_2, called *deficiency*:

$$\mathrm{nr}(L_1, L_2) = \mathrm{nr}_{k,k}(L_1, L_2) + \mathrm{nr}_{1,k}(L_1, L_2) + \mathrm{nr}^s_{1,k}(L_1, L_2) \quad \text{where} \tag{2}$$

- $\mathrm{nr}_{k,k}(L_1, L_2)$ is the number of pairs of k-sets not reversed in L_1, L_2,
- $\mathrm{nr}_{1,k}(L_1, L_2)$ is the number of reversible nonsimple pairs $\{S, u\}$ not reversed in L_1, L_2,
- $\mathrm{nr}^s_{1,k}(L_1, L_2)$ is the number of reversible simple pairs $\{S, u\}$ not reversed in L_1, L_2.

The following equality follows directly from the definitions.

Proposition 2. *For every permutation σ of $[n]$, $1 < k \leq n$, $L_1 \in \mathcal{L}_n^k(id)$, $L_2 \in \mathcal{L}_n^k(\sigma)$ it holds that*

$$d(L_1, L_2) = \binom{\binom{n}{k}}{2} + 2\binom{n}{k+1} + \mathrm{inv}(\sigma) - s(\sigma) - \mathrm{nr}(L_1, L_2).$$

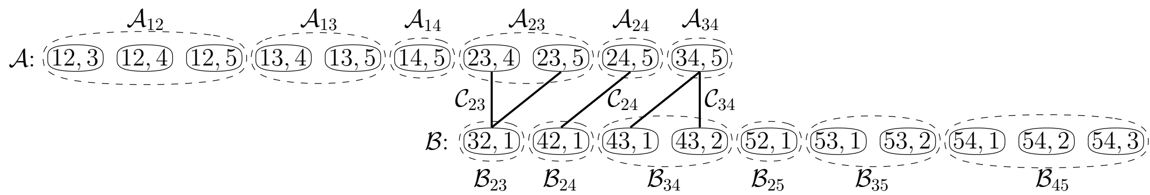

Figure 1: The dependency graph $G_5^2(\overline{id})$. The empty sets $\mathcal{A}_S$, $\mathcal{B}_S$ are not depicted.

3 Dependency graphs

In this section we define a dependency graph for a given permutation σ of $[n]$ and show that its vertex cover number bounds from below the deficiency of two linear extensions $L_1 \in \mathcal{L}_n^k(id)$ and $L_2 \in \mathcal{L}_n^k(\sigma)$ plus the number of simple pairs with respect to σ.

Let us fix $1 < k \le n$ and a permutation σ of $[n]$. For a k-set S let $\mathcal{A}_S (= \mathcal{A}_S^{id})$ and $\mathcal{B}_S^\sigma$ be the families of all pairs $\{S, u\}$ that are free in $\mathcal{L}(id)$, respectively in $\mathcal{L}(\sigma)$. That is,

$$\mathcal{A}_S = \{\{S, u\};\ u >_{id} v \text{ for every } v \in S\}, \quad \mathcal{B}_S^\sigma = \{\{S, u\};\ u >_\sigma v \text{ for every } v \in S\}. \tag{3}$$

Note that $\mathcal{A}_S$ and $\mathcal{B}_S^\sigma$ can be both empty as

$$|\mathcal{A}_S| = n - \max_{id}(S), \quad |\mathcal{B}_S^\sigma| = n - \sigma^{-1}(\max_\sigma(S)). \tag{4}$$

Let $\mathcal{C}_S^\sigma$ be the complete bipartite graph on vertex sets $\mathcal{A}_S$ and $\mathcal{B}_S^\sigma$ as bipartite classes. The index σ in $\mathcal{B}_S^\sigma$ and $\mathcal{C}_S^\sigma$ is omitted whenever it is clear from the context. Note our abuse of notation since $\mathcal{A}_S \cap \mathcal{B}_S$ may be nonempty, but when $\mathcal{A}_S$ and $\mathcal{B}_S$ are regarded as sets of vertices of $\mathcal{C}_S$, they are considered to be disjoint. More precisely, $\{S, u\} \in \mathcal{A}_S \cap \mathcal{B}_S$ if and only if $\{S, u\}$ is simple, but we distinguish the copies of a simple pair $\{S, u\}$ as distinct vertices in $\mathcal{A}_S$ and $\mathcal{B}_S$. The edge between these copies is called a *simple edge*.

Let $\mathcal{A} = \bigcup_{S \in \binom{[n]}{k}} \mathcal{A}_S$ and $\mathcal{B} = \bigcup_{S \in \binom{[n]}{k}} \mathcal{B}_S$. The *dependency graph* $G_n^k(\sigma)$ of σ is a (bipartite) graph on vertices of $\mathcal{A}$ and $\mathcal{B}$ defined by

$$G_n^k(\sigma) = \bigcup_{S \in \binom{[n]}{k}} \mathcal{C}_S^\sigma.$$

Note that the indices k and n may be omitted whenever they are clear from the context. See Figure 1 for an example of the dependency graph $G_5^2(\overline{id})$.

Note that $G(\sigma)$ has no edge if and only if $\mathcal{A}_S = \emptyset$ or $\mathcal{B}_S = \emptyset$ for every k-set $S \subseteq [n]$. Since $\mathcal{A}_S = \emptyset$ if and only if $n \in S$ and $\mathcal{B}_S = \emptyset$ if and only if $\sigma(n) \in S$ by (4), this is equivalent to $k = n$ or, $k = n - 1$ and $\sigma(n) \ne n$.

The edges of $G(\sigma)$ have the following interpretation, called *dependency*.

Proposition 3. *Let σ be a permutation of $[n]$, $1 < k \le n$, $L_1 \in \mathcal{L}_n^k(id)$, and $L_2 \in \mathcal{L}_n^k(\sigma)$. For every edge of $G_n^k(\sigma)$ between $\{S, u\} \in \mathcal{A}_S$ and $\{S, v\} \in \mathcal{B}_S$, if there is a k-set T containing u and v such that $\{S, T\}$ is reversed in L_1, L_2, then $u <_{L_1} S$ or $v <_{L_2} S$.*

Proof. Suppose that $\{S, T\}$ is reversed, $S <_{L_1} u$ and $S <_{L_2} v$. Since $u <_{L_1} T$ and $v <_{L_2} T$, it follows that S is before T in both L_1 and L_2, a contradiction. □

Recall that a *vertex cover* of a graph G is a set of vertices covering every edge of G. The *vertex cover number* $\alpha(G)$ of the graph G is the minimum size of a vertex cover of G. In the first part towards the proof of Theorem 1 we show that the vertex cover number of $G_n^k(\sigma)$ bounds from below the deficiency of two linear extensions $L_1 \in \mathcal{L}_n^k(id)$, $L_2 \in \mathcal{L}_n^k(\sigma)$ plus the number of simple pairs with respect to σ.

Lemma 4. *For every permutation σ of $[n]$, $1 < k \leq n$, $L_1 \in \mathcal{L}_n^k(id)$, $L_2 \in \mathcal{L}_n^k(\sigma)$, the dependency graph $G_n^k(\sigma)$ has a vertex cover of size at most* $\mathrm{nr}(L_1, L_2) + s(\sigma)$.

Proof. We start with the following sets of vertices of $G(\sigma)$ (regarded as disjoint):

$$V_{\mathcal{A}} = \{\{S, u\} \in \mathcal{A};\ u <_{L_1} S\},\quad V_{\mathcal{B}} = \{\{S, u\} \in \mathcal{B};\ u <_{L_2} S\}.$$

If $\{S, u\} \in \mathcal{A}$ is nonsimple, then it is fixed in $\mathcal{L}(\sigma)$, so $u <_{L_2} S$. Similarly, if $\{S, u\} \in \mathcal{B}$ is nonsimple, then it is fixed in $\mathcal{L}(id)$, so $u <_{L_1} S$. Hence,

$$|\{\{S, u\} \in V_{\mathcal{A}};\ \{S, u\} \text{ nonsimple}\}| + |\{\{S, u\} \in V_{\mathcal{B}};\ \{S, u\} \text{ nonsimple}\}| = \mathrm{nr}_{1,k}(L_1, L_2). \quad (5)$$

To ensure that every simple edge is covered, we put

$$V'_{\mathcal{A}} = V_{\mathcal{A}} \cup \{\{S, u\} \in \mathcal{A};\ \{S, u\} \text{ is simple}, \{S, u\} \notin V_{\mathcal{A}} \text{ and } \{S, u\} \notin V_{\mathcal{B}}\}.$$

If a simple $\{S, u\}$ belongs to both $V'_{\mathcal{A}}$ and $V_{\mathcal{B}}$, then it is not reversed in L_1, L_2. Hence,

$$|\{\{S, u\} \in V'_{\mathcal{A}};\ \{S, u\} \text{ simple}\}| + |\{\{S, u\} \in V_{\mathcal{B}};\ \{S, u\} \text{ simple}\}| \leq \mathrm{nr}^s_{1,k}(L_1, L_2) + s(\sigma). \quad (6)$$

Now, which edges of $G(\sigma)$ are covered by $V'_{\mathcal{A}} \cup V_{\mathcal{B}}$?

a) Every simple edge is covered.

b) By Proposition 3, an edge between $\{S, u\} \in \mathcal{A}$ and $\{S, v\} \in \mathcal{B}$ is covered if there is a k-set T containing u and v such that $\{S, T\}$ is reversed.

c) For simple pairs $\{S, u\}$, $\{S, v\}$ with $u \neq v$, at least one of the two edges between $\{S, u\} \in \mathcal{A}$ and $\{S, v\} \in \mathcal{B}$, and between $\{S, v\} \in \mathcal{A}$ and $\{S, u\} \in \mathcal{B}$ is covered because these four vertices induce $K_{2,2}$ in $\mathcal{C}_S$ and both its simple edges are covered by a).

Therefore, for every uncovered edge between $\{S, u\} \in \mathcal{A}$ and $\{S, v\} \in \mathcal{B}$ we may assign a k-set T containing u and v such that $\{S, T\}$ is not reversed. Specifically, we put $T = S' \cup \{u, v\}$ where $S' \subseteq S$ consists of the $k - 2$ least elements in S. Note that the pair $\{S, T\}$ cannot be assigned to any other uncovered edge. This implies that the number of edges uncovered by $V'_{\mathcal{A}} \cup V_{\mathcal{B}}$ is at most $\mathrm{nr}_{k,k}(L_1, L_2)$.

It remains to add the vertex from $\mathcal{B}$ of each uncovered edge to $V_{\mathcal{B}}$, so we get a new set $V'_{\mathcal{B}}$. Finally, from (2), (5), (6) we conclude that the vertices of $V'_{\mathcal{A}}$ with $V'_{\mathcal{B}}$ form a vertex cover of $G(\sigma)$ of size at most $\mathrm{nr}(L_1, L_2) + s(\sigma)$. □

From Proposition 2 and Lemma 4 we obtain the following upper bound.

Corollary 5. *For every permutation σ of $[n]$, $1 < k \leq n$, $L_1 \in \mathcal{L}_n^k(id)$, and $L_2 \in \mathcal{L}_n^k(\sigma)$ it holds that*

$$d(L_1, L_2) \leq \binom{\binom{n}{k}}{2} + 2\binom{n}{k+1} + \mathrm{inv}(\sigma) - \alpha(G_n^k(\sigma)).$$

4 Tight construction

In this section we finish the proof of Theorem 1. The next lemma shows that the bound from Corollary 5 is attained by some pair of linear extensions $L_1 \in \mathcal{L}_n^k(id)$, $L_2 \in \mathcal{L}_n^k(\sigma)$. Hence, it gives us the maximal distance between two linear extensions with fixed order of atoms. Then we show that maximum over all permutations σ of $[n]$ is attained only by $\sigma = \overline{id}$. Finally, we give an explicit formula for $\alpha(G_n^k(\overline{id}))$.

For a linear extension L of $\mathcal{B}_n^{1,k}$ and $i \in [n]$, the set of positions in L between the ith atom (in the order $<_L$) and the next atom is called the *ith slot*. The *last slot* of L is the nth slot; that is, the set of positions after the last atom in L.

Lemma 6. *For every permutation σ of $[n]$, $1 < k \leq n$, there exists $L_1 \in \mathcal{L}_n^k(id)$, $L_2 \in \mathcal{L}_n^k(\sigma)$ such that*

$$d(L_1, L_2) = \binom{\binom{n}{k}}{2} + 2\binom{n}{k+1} + \operatorname{inv}(\sigma) - \alpha(G_n^k(\sigma)).$$

Proof. By the definition of $G(\sigma)$ we have

$$\alpha(G(\sigma)) = \sum_{S \in \binom{[n]}{k}} \min(|\mathcal{A}_S|, |\mathcal{B}_S|).$$

To construct the desired extensions $L_1 \in \mathcal{L}(id)$, $L_2 \in \mathcal{L}(\sigma)$, we first decide for each k-set S into which slot it is placed in L_1 and in L_2. Our aim is to reverse all free pairs $\{S, u\}$ in a larger bipartite class of $\mathcal{C}_S$. (If $|\mathcal{A}_S| = |\mathcal{B}_S|$ we choose the class $\mathcal{A}_S$.) For this purpose, we put S into the smallest slot possible in one extension and into the last slot of the other extension. Namely, for

$$i = \max_{id}(S) = n - |\mathcal{A}_S|, \quad j = \sigma^{-1}(\max_\sigma(S)) = n - |\mathcal{B}_S|,$$

- if $i \leq j$, we put S into the ith slot in L_1 and into the last slot in L_2;
- if $i > j$, we put S into the jth slot in L_2 and into the last slot in L_1.

Now, if $|\mathcal{A}_S| \geq |\mathcal{B}_S|$, then every pair $\{S, u\} \in \mathcal{A}_S$ is reversed since $S <_{L_1} u$ and $u <_{L_2} S$, and every nonsimple pair $\{S, v\} \in \mathcal{B}_S$ is not reversed since $v <_{L_1} S$ as $\{S, v\}$ is fixed in $\mathcal{L}(id)$ and $v <_{L_2} S$. The case $|\mathcal{A}_S| < |\mathcal{B}_S|$ is analogous. In either way, all simple pairs are reversed, so $\operatorname{nr}_{1,k}^s(L_1, L_2) = 0$. Recall that $\{S, u\}$ is simple if and only if $\{S, u\} \in \mathcal{A}_S \cap \mathcal{B}_S$. Hence, the number of simple pairs plus nonsimple not reversed pairs $\{S, u\}$ for some u is precisely $\min(|\mathcal{A}_S|, |\mathcal{B}_S|)$. Altogether for all k-sets S,

$$\operatorname{nr}_{1,k}(L_1, L_2) + s(\sigma) = \alpha(G(\sigma)). \tag{7}$$

It remains to show that by a proper ordering within slots of L_1, L_2, we reverse all pairs of k-sets.

1. If distinct k-sets are not together in a last slot of the same linear extension, then they are already reversed. Thus, in all but last slots we may choose an arbitrary order.
2. Let $\mathcal{Z}$ be the family of all k-sets that are in last slots of both linear extensions; that is,

$$\mathcal{Z} = \left\{ S \in \binom{[n]}{k};\ \{n = \max_{id}([n]), \max_\sigma([n])\} \subseteq S \right\},$$

and let $\mathcal{Z}_1$ and $\mathcal{Z}_2$ be the families of remaining k-sets in the last slots of L_1 and L_2, respectively. First, we arbitrarily order $\mathcal{Z}$ in the last slot of L_1 so that $\mathcal{Z}$ precedes $\mathcal{Z}_1$. Then we order $\mathcal{Z}$ reversely (than in L_1) in the last slot of L_2 so that $\mathcal{Z}$ precedes $\mathcal{Z}_2$. See the example below.

3. Finally, we order $\mathcal{Z}_1$ in the last slots of L_1 reversely than it appears in L_2. Analogously, we order $\mathcal{Z}_2$ in the last slots of L_2 reversely than it appears in L_1.

Since $\mathrm{nr}_{k,k}(L_1, L_2) = 0$ and $\mathrm{nr}^s_{1,k}(L_1, L_2) = 0$, it follows from Proposition 2 that

$$d(L_1, L_2) = \binom{\binom{n}{k}}{2} + 2\binom{n}{k+1} + \mathrm{inv}(\sigma) - s(\sigma) - \mathrm{nr}_{1,k}(L_1, L_2)$$

so the statement follows from (7). □

Note that the constructed linear extensions are not unique. For example, for $\sigma = 2341$ and $k = 2$ we have $\mathcal{Z} = \{14\}$, $\mathcal{Z}_1 = \{23, 24, 34\}$, $\mathcal{Z}_2 = \{12, 13\}$ and the above construction yields to pairs $L_1 = (1, 2, 12, 3, 13, 4, 14, 34, 24, 23)$, $L_2 = (2, 3, 23, 4, 24, 34, 1, 14, 13, 12)$ or $L_1 = (1, 2, 12, 3, 13, 4, 14, 24, 34, 23)$, $L_2 = (2, 3, 23, 4, 34, 24, 1, 14, 13, 12)$.

By Corollary 5 and Lemma 6, to determine the linear extension diameter of $\mathcal{B}^{1,k}_n$ it only remains to find the maximal value of $\mathrm{inv}(\sigma) - \alpha(G^k_n(\sigma))$ over all permutations σ of $[n]$. The last piece of proof of Theorem 1 shows that the maximum is attained by the reversed identity.

Lemma 7. *For every $1 < k \leq n$, $\mathrm{inv}(\sigma) - \alpha(G^k_n(\sigma))$ is maximized (only) for $\sigma = \overline{id}$.*

Proof. For every permutation σ of $[n]$ there is a sequence $\sigma = \sigma_1, \sigma_2, \ldots, \sigma_p = \overline{id}$ of permutations such that σ_i and σ_{i+1} differ by an adjacent transposition and $\mathrm{inv}(\sigma_{i+1}) = \mathrm{inv}(\sigma_i) + 1$. It suffices to show that $\alpha(G(\sigma_{i+1})) \leq \alpha(G(\sigma_i))$ for every $1 \leq i < p$.

Let $x < y$ be the two adjacent atoms in which σ_i and σ_{i+1} differ. Since $\mathrm{inv}(\sigma_{i+1}) = \mathrm{inv}(\sigma_i) + 1$, we have $x <_{\sigma_i} y$ and $y <_{\sigma_{i+1}} x$. Recall that $|\mathcal{A}_S| = n - \max_{id}(S)$, $|\mathcal{B}^\sigma_S| = n - \sigma^{-1}(\max_\sigma(S))$ and

$$\alpha(G(\sigma)) = \sum_{S \in \binom{[n]}{k}} \min(|\mathcal{A}_S|, |\mathcal{B}^\sigma_S|).$$

By the definition (3) of $\mathcal{B}^\sigma_S$, for every k-set S it holds that

$$\mathcal{B}^{\sigma_{i+1}}_S = \begin{cases} \mathcal{B}^{\sigma_i}_S \cup \{\{S, x\}\} & \text{if } x \notin S \text{ and } y = \max_{\sigma_i}(S), \\ \mathcal{B}^{\sigma_i}_S \setminus \{\{S, y\}\} & \text{if } x = \max_{\sigma_i}(S) \text{ and } y \notin S, \\ \mathcal{B}^{\sigma_i}_S & \text{otherwise.} \end{cases} \tag{8}$$

It follows that $\alpha(\mathcal{C}^{\sigma_{i+1}}_S) = \min(|\mathcal{A}_S|, |\mathcal{B}^{\sigma_{i+1}}_S|)$ differs from $\alpha(\mathcal{C}^{\sigma_i}_S)$ by at most 1. To conclude the proof, we claim for every k-set S with $x \notin S$ and $y = \max_{\sigma_i}(S)$ that

$$\text{if } \alpha(\mathcal{C}^{\sigma_{i+1}}_S) = \alpha(\mathcal{C}^{\sigma_i}_S) + 1, \text{ then } \alpha(\mathcal{C}^{\sigma_{i+1}}_T) = \alpha(\mathcal{C}^{\sigma_i}_T) - 1 \text{ for the } k\text{-set } T = (S \cup \{x\}) \setminus \{y\}; \tag{9}$$

that is, an increase of the minimum vertex cover in one component is compensated by a decrease in another (unique) component. From (9) it follows that $\alpha(G(\sigma_{i+1})) \leq \alpha(G(\sigma_i))$.

To prove the claim (9), assume that $\alpha(\mathcal{C}^{\sigma_{i+1}}_S) = \alpha(\mathcal{C}^{\sigma_i}_S) + 1$ for a k-set S with $x \notin S$ and $y = \max_{\sigma_i}(S)$. Note that $x = \max_{\sigma_i}(T)$ and $y \notin T$ for the k-set $T = (S \cup \{x\}) \setminus \{y\}$, so $|\mathcal{B}^{\sigma_{i+1}}_S| = |\mathcal{B}^{\sigma_i}_S| + 1$ and $|\mathcal{B}^{\sigma_{i+1}}_T| = |\mathcal{B}^{\sigma_i}_T| - 1$ by (8). Thus, we have

$$|\mathcal{B}^{\sigma_{i+1}}_T| = |\mathcal{B}^{\sigma_i}_S| < |\mathcal{A}_S| = n - \max(S) \leq n - \max(T) = |\mathcal{A}_T|.$$

The first equality holds since $\sigma_i^{-1}(y) = \sigma_{i+1}^{-1}(x)$, the strict inequality holds since $\mathcal{B}_S^{\sigma_i}$ is the smaller class of $\mathcal{C}_S^{\sigma_i}$, the last inequality holds since $x < y$. Hence, $\mathcal{B}_T^{\sigma_{i+1}}$ is the smaller class of $\mathcal{C}_T^{\sigma_{i+1}}$, and therefore (9) holds. □

From Corollary 5, Lemma 6 and Lemma 7 we obtain the exact value of the linear extension diameter of $\mathcal{B}_n^{1,k}$. Let us write shortly $\alpha_n^k = \alpha(G_n^k(\overline{id}))$.

Corollary 8. *For every* $1 < k \leq n$,

$$\mathrm{led}(\mathcal{B}_n^{1,k}) = \binom{\binom{n}{k}}{2} + 2\binom{n}{k+1} + \binom{n}{2} - \alpha_n^k.$$

That is, $\mathrm{led}(\mathcal{B}_n^{1,k}) = \mathrm{inc}(\overline{id}) - \alpha_n^k$. The value of α_n^k can be expressed recursively as follows.

Lemma 9. *For every* $1 < k \leq n$, *it holds* $\alpha_k^k = \alpha_{k+1}^k = 0$ *and* $\alpha_{n+2}^k = \alpha_n^k + \binom{n}{k}$. *Hence,*

$$\alpha_n^k = \sum_{\substack{i=k \\ i \equiv n \pmod 2}}^{n-2} \binom{i}{k}.$$

Proof. $G_k^k(\overline{id})$ is empty and $G_{k+1}^k(\overline{id})$ is edgeless, so $\alpha_k^k = \alpha_{k+1}^k = 0$. In order to calculate the difference $\alpha_{n+2}^k - \alpha_n^k$, let us assume that the ground set $[n]$ is extended by elements 0 and $n+1$. We consider the graph $\mathcal{C}_\mathcal{S}$ in $G_{n+2}^k(\overline{id})$ for each k-set $S \in \binom{\{0,\ldots,n+1\}}{k}$. If S contains 0 or $n+1$, then $\mathcal{B}_S$ or $\mathcal{A}_S$, respectively, is empty.

Otherwise; $S \in \binom{[n]}{k}$, the class $\mathcal{A}_S$ is extended by the vertex $\{S, n+1\}$, and the class $\mathcal{B}_S$ is extended by the vertex $\{S, 0\}$. Thus, the minimal vertex cover of $\mathcal{C}_S$ increments exactly by 1 in $G_{n+2}^k(\overline{id})$ compared to $G_n^k(\overline{id})$. Hence we establish the recurrence $\alpha_{n+2}^k - \alpha_n^k = \binom{n}{k}$, which leads to the explicit formula. □

Now, Theorem 1 follows from Corollary 8 and Lemma 9. For $k = 2, 3, 4$ we obtain the following formulas for α_n^k and $led(\mathcal{B}_n^{1,k})$.

k	α_n^k $led(\mathcal{B}_n^{1,k})$
2	$\frac{1}{48}(4n^3 - 18n^2 + 20n - 3 + 3(-1)^n)$ $\frac{1}{16}(2n^4 - 4n^2 + 1 - (-1)^n)$
3	$\frac{1}{96}(2n^4 - 16n^3 + 40n^2 - 32n + 3 - 3(-1)^n)$ $\frac{1}{288}(4n^6 - 24n^5 + 70n^4 - 168n^3 + 376n^2 - 240n - 9 + 9(-1)^n)$
4	$\frac{1}{960}(4n^5 - 50n^4 + 220n^3 - 400n^2 + 256n - 15 + 15(-1)^n)$ $\frac{1}{5760}(5n^8 - 60n^7 + 290n^6 - 648n^5 + 185n^4 + 2100n^3 - 660n^2 - 1392n + 90 - 90(-1)^n)$

5 Diametral linear extensions

Recall that for diametral linear extensions L_1, L_2 of $\mathcal{B}_n^{1,k}$ we may assume $L_1 \in \mathcal{L}_n^k(id)$. All other diametral pairs are obtained by automorphisms of $\mathcal{B}_n^{1,k}$. From previous sections we know that linear extensions $L_1 \in \mathcal{L}_n^k(id)$ and L_2 of $\mathcal{B}_n^{1,k}$ are diametral if and only if $L_2 \in \mathcal{L}_n^k(\overline{id})$ and

$\mathrm{nr}(L_1, L_2) = \alpha_n^k$. In this section we describe an explicit construction of all diametral pairs of linear extensions, see Theorem 17. The case $k = 2$ is stated separately in Theorem 19.

We start with two additional properties of dependency graphs, in particular of $G_n^k(\overline{id})$.

Proposition 10. *Let σ be a permutation of $[n]$, $1 < k \le n$. If there is an edge between $\{S, u\}$ and $\{S, v\}$ in $G_n^k(\sigma)$, then no vertex $\{T, x\}$ with $x \in S$ and T containing u and v is in $G_n^k(\sigma)$.*

Proof. Assume that $\{S, u\} \in \mathcal{A}_S$ and $\{S, v\} \in \mathcal{B}_S$; that is, $u >_{id} x$ and $v >_\sigma x$ for every $x \in S$. Hence by (3), $\{T, x\} \notin \mathcal{A}_T$ if T contains u, and $\{T, x\} \notin \mathcal{B}_T$ if T contains v. □

To each edge of $G_n^k(\overline{id})$ where $1 < k \le n$ we assign the following pairs of k-sets called a *dependency family.* Assume that an edge e joins vertices $\{S, u\}$ and $\{S, v\}$, then

$$\mathcal{D}(e) = \{\{S, S' \cup \{u, v\}\};\ S' \subset S, |S'| = k - 2\}. \tag{10}$$

By Proposition 3, for every edge e of $G_n^k(\overline{id})$ between $\{S, u\} \in \mathcal{A}_S$ and $\{S, v\} \in \mathcal{B}_S$, if $S <_{L_1} u$ and $S <_{L_2} v$ for linear extensions $L_1 \in \mathcal{L}_n^k(id)$, $L_2 \in \mathcal{L}_n^k(\overline{id})$, then every pair of k-sets in $\mathcal{D}(e)$ is unreversed in L_1, L_2. Another key property of dependency families is as follows.

Proposition 11. *The dependency families of edges of $G_n^k(\overline{id})$ where $1 < k \le n$ are pairwise disjoint.*

Proof. First, consider distinct edges $e = \{\{S, u\}, \{S, v\}\}$, $e' = \{\{S, u'\}, \{S, v'\}\}$ from the same component $\mathcal{C}_S$ of $G_n^k(\overline{id})$. Since there are no simple pairs for $\sigma = \overline{id}$, we have $\{u, v\} \ne \{u', v'\}$. Moreover, none of u, u', v, v' is in S. Hence it follows from (10) that $\mathcal{D}(e)$ and $\mathcal{D}(e')$ are disjoint.

Second, suppose that $\{S, T\} \in \mathcal{D}(e) \cap \mathcal{D}(e')$ for some edges $e = \{\{S, u\}, \{S, v\}\}$, $e' = \{\{T, u'\}, \{T, v'\}\}$ from distinct components $\mathcal{C}_S$ and $\mathcal{C}_T$. Then by (10), $\{u, v\} \subseteq T$ and $\{u', v'\} \subseteq S$ which contradicts Proposition 10. □

A *vertex-edge cover* of a graph $G = (V, E)$ is a pair (A, B) where $A \subseteq V$, $B \subseteq E$ such that every edge not in B has a vertex in A. Its size is $|A| + |B|$. A minimal vertex-edge cover is a vertex-edge cover of minimal size. Observe that complete bipartite graphs are more efficiently covered by vertices than edges, except $K_{1,1}$ which has a minimal cover by a single vertex or by a single edge.

Observation 12. *Every minimal vertex-edge cover of $K_{n,m}$ where $n, m \ge 1$ is a vertex cover unless $n = m = 1$.*

We have seen in the proof of Lemma 4 that unreversed pairs that contribute to $\mathrm{nr}(L_1, L_2)$ are related to vertex-edge covers of $G_n^k(\sigma)$. For precise description we need to consider vertex-edge covers of $G_n^k(\overline{id})$ more carefully for the case $k = 2$.

Observation 13. *$G_2^2(\overline{id})$ is empty, $G_3^2(\overline{id})$ is edgeless, and for $n \ge 4$ the only component of $G_n^2(\overline{id})$ isomorphic to $K_{1,1}$, is the component $\mathcal{C}_{2(n-1)}$.*

The next lemma shows that diametral linear extensions have all pairs of k-sets reversed, up to one exception for $k = 2$.

Lemma 14. *Let $1 < k \le n$, $L_1 \in \mathcal{L}_n^k(id)$, $L_2 \in \mathcal{L}_n^k(\overline{id})$ such that a pair $\{A, B\}$ of k-sets is not reversed in L_1, L_2. Then $\mathrm{nr}(L_1, L_2) > \alpha_n^k$ unless $k = 2$, $n \ge 4$, and $\{A, B\} = \{2(n-1), 1n\}$.*

Proof. First, recall that for $\sigma = \overline{id}$ there are no simple pairs. Similarly as in (5), for (disjoint) sets of vertices

$$V_{\mathcal{A}} = \{\{S,u\} \in \mathcal{A};\ u <_{L_1} S\},\quad V_{\mathcal{B}} = \{\{S,u\} \in \mathcal{B};\ u <_{L_2} S\}$$

we have $|V_{\mathcal{A}}| + |V_{\mathcal{B}}| = \mathrm{nr}_{1,k}(L_1, L_2)$. If $V_{\mathcal{A}} \cup V_{\mathcal{B}}$ covers $G(\overline{id})$, then $|V_{\mathcal{A}}| + |V_{\mathcal{B}}| \geq \alpha_n^k$. Since $\mathrm{nr}_{k,k}(L_1, L_2) \geq 1$ by the assumption of the lemma, we obtain

$$\mathrm{nr}(L_1, L_2) = \mathrm{nr}_{k,k}(L_1, L_2) + \mathrm{nr}_{1,k}(L_1, L_2) > \alpha_n^k.$$

Otherwise, let $U \neq \emptyset$ be the set of edges of $G(\overline{id})$ uncovered by $V_{\mathcal{A}} \cup V_{\mathcal{B}}$. Since every component of $G(\overline{id})$ is a complete bipartite graph, $(V_{\mathcal{A}} \cup V_{\mathcal{B}}, U)$ is a vertex-edge cover of $G(\overline{id})$ and α_n^k is the minimal size of a vertex cover of $G(\overline{id})$, we have $|V_{\mathcal{A}}| + |V_{\mathcal{B}}| + |U| \geq \alpha_n^k$ by Observation 12. By Proposition 3, every pair of k-sets in the dependency family $\mathcal{D}(e)$ of each edge e from U is not reversed in L_1, L_2. Since dependency families are disjoint by Proposition 11, we have

$$\mathrm{nr}_{k,k}(L_1, L_2) \geq \sum_{e \in U} |\mathcal{D}(e)|.$$

We distinguish two cases regarding k. If $k \geq 3$, then $|\mathcal{D}(e)| \geq 3$ for every edge e by (10). Consequently, $\mathrm{nr}_{k,k}(L_1, L_2) \geq 3|U| > |U|$ since $U \neq \emptyset$. Therefore,

$$\mathrm{nr}(L_1, L_2) = \mathrm{nr}_{1,k}(L_1, L_2) + \mathrm{nr}_{k,k}(L_1, L_2) > |V_{\mathcal{A}}| + |V_{\mathcal{B}}| + |U| \geq \alpha_n^k.$$

If $k = 2$, then $|\mathcal{D}(e)| = 1$ for every edge e, so $\mathrm{nr}_{k,k}(L_1, L_2) \geq |U|$. If there is an uncovered edge $e \in U$ in some component $\mathcal{C}_S \not\simeq K_{1,1}$, then by Observation 12, the restriction of the vertex-edge cover $(V_{\mathcal{A}} \cup V_{\mathcal{B}}, U)$ on the component $\mathcal{C}_S$ is of size greater than $\alpha(\mathcal{C}_S)$. Consequently,

$$\mathrm{nr}(L_1, L_2) = \mathrm{nr}_{1,k}(L_1, L_2) + \mathrm{nr}_{k,k}(L_1, L_2) \geq |V_{\mathcal{A}}| + |V_{\mathcal{B}}| + |U| > \alpha_n^k.$$

By Observation 13, the only remaining case is when U contains exactly the edge e' from the component $\mathcal{C}_{2(n-1)} \simeq K_{1,1}$. Since $\mathcal{D}(e') = \{\{2(n-1), 1n\}\}$, this is the exceptional case in the statement. □

Every minimal vertex cover of $G_n^k(\overline{id})$ consists of minimal bipartite classes of each component $\mathcal{C}_S$. Note that if $|\mathcal{A}_S| = |\mathcal{B}_S| \neq 0$ for some k-set S, a minimal vertex cover of $G_n^k(\overline{id})$ is not unique. Let V be a minimal vertex cover of $G_n^k(\overline{id})$. Two linear extensions $L_1 \in \mathcal{L}_n^k(id)$, $L_2 \in \mathcal{L}_n^k(\overline{id})$ are said to be V-*compatible* if for every k-set S,

a) S is in the ith slot in L_1 and in the last slot in L_2 if $\mathcal{B}_S \subseteq V$,

b) S is in the jth slot in L_2 and in the last slot in L_1 if $\mathcal{A}_S \subseteq V$,

c) all pairs of k-sets are reversed in L_1, L_2,

where $i = \max(S) = n - |\mathcal{A}_S|$, $j = n - \min(S) + 1 = n - |\mathcal{B}_S|$. Note that if $\mathcal{A}_S = \mathcal{B}_S = \emptyset$, then $i = j = n$, so the slots for S are correctly defined. An alternative definition of V-compatible extensions is as follows.

Proposition 15. *Let $1 < k \leq n$ and V be a minimal vertex cover of $G_n^k(\overline{id})$. Then $L_1 \in \mathcal{L}_n^k(id)$, $L_2 \in \mathcal{L}_n^k(\overline{id})$ are V-compatible if and only if $V_\mathcal{A} \cup V_\mathcal{B} = V$ and $\mathrm{nr}_{k,k}(L_1, L_2) = 0$ where*

$$V_\mathcal{A} = \{\{S, u\} \in \mathcal{A};\ u <_{L_1} S\}, \quad V_\mathcal{B} = \{\{S, u\} \in \mathcal{B};\ u <_{L_2} S\}. \tag{11}$$

Proof. Assume $V_\mathcal{A} \cup V_\mathcal{B} = V$. Thus, for every k-set S, $\mathcal{A}_S \cap V_\mathcal{A} = \emptyset$ and $\mathcal{B}_S \subseteq V_\mathcal{B}$ if $\mathcal{B}_S \subseteq V$, and $\mathcal{B}_S \cap V_\mathcal{B} = \emptyset$ and $\mathcal{A}_S \subseteq V_\mathcal{A}$ if $\mathcal{A}_S \subseteq V$. Equivalently by (11),

- $S <_{L_1} u$ for every $\{S, u\} \in \mathcal{A}_S$ and $u <_{L_2} S$ for every $\{S, u\} \in \mathcal{B}_S$ if $\mathcal{B}_S \subseteq V$,
- $S <_{L_2} u$ for every $\{S, u\} \in \mathcal{B}_S$ and $u <_{L_1} S$ for every $\{S, u\} \in \mathcal{A}_S$ if $\mathcal{A}_S \subseteq V$.

In particular,

- $\max(S) <_{L_1} S <_{L_1} \max(S) + 1$ and $1 <_{L_2} S$ if $\mathcal{B}_S \subseteq V$,
- $\min(S) <_{L_2} S <_{L_2} \min(S) - 1$ and $n <_{L_1} S$ if $\mathcal{A}_S \subseteq V$.

That is, a) and b) holds. Since $\mathrm{nr}_{k,k}(L_1, L_2) = 0$ is equivalent to c), we obtain that L_1, L_2 are V-compatible. The other implication follows straightforwardly from the definition. □

An example of V-compatible linear extensions for the minimal vertex cover V with $\mathcal{B}_S \subseteq V$ whenever $|\mathcal{A}_S| = |\mathcal{B}_S|$ is constructed in Lemma 6. For $n = 4$ we obtain for instance $L_1 = (1, 2, 12, 3, 13, 23, 4, 14, 24, 34)$ and $L_2 = (4, 3, 34, 2, 24, 1, 14, 23, 13, 12)$.

For two linear extensions L_1, L_2 of $B_n^{1,k}$ let $\mathcal{Z}(L_1, L_2)$, $\mathcal{Z}_1(L_1, L_2)$, and $\mathcal{Z}_2(L_1, L_2)$ be the families of k-sets that are in the last slots both in L_1 and L_2, in the last slot only in L_1, and in the last slot only in L_2, respectively.

Proposition 16. *Let L_1, L_2 be linear extensions of $B_n^{1,k}$, $1 < k \leq n$. If $\mathrm{nr}_{k,k}(L_1, L_2) = 0$ then $S <_{L_1} T_1$ and $S <_{L_2} T_2$ for every $S \in \mathcal{Z}(L_1, L_2)$, $T_1 \in \mathcal{Z}_1(L_1, L_2)$, and $T_2 \in \mathcal{Z}_2(L_1, L_2)$.*

Proof. We have $T_1 <_{L_2} S$ and $T_2 <_{L_1} S$ since T_1 is not in the last slot of L_2 and T_2 is not in the last slot of T_1. Since both $\{S, T_1\}$ and $\{S, T_2\}$ are reversed, the statement follows. □

It follows that all V-compatible pairs L_1, L_2 for a given minimal vertex cover V of $G_n^k(\overline{id})$ can be obtained by ordering k-sets in slots as described by steps 1.– 3. in the construction in Lemma 6 for $\sigma = \overline{id}$. Note that there are two degrees of freedom in the construction:

i) the order of k-sets in each but last slot in L_1 or L_2,

ii) the order of $\mathcal{Z}(L_1, L_2)$ in the last slot of L_1.

Theorem 17. *Linear extensions $L_1 \in \mathcal{L}_n^k(id)$, L_2 of $B_n^{1,k}$ for $2 < k \leq n$ are diametral if and only if they are V-compatible for some minimal vertex cover V of the dependency graph $G_n^k(\overline{id})$. All other diametral linear extensions are obtained by automorphisms of $B_n^{1,k}$.*

Proof. The sufficiency follows from $\mathrm{nr}(L_1, L_2) = \alpha_n^k$ and $L_2 \in \mathcal{L}(\overline{id})$ for every V-compatible L_1, L_2 since $\mathrm{nr}_{1,k}(L_1, L_2) = |V_\mathcal{A} \cup V_\mathcal{B}| = |V| = \alpha_n^k$ by Proposition 15 and $\mathrm{nr}_{k,k}(L_1, L_2) = 0$ by c). The necessity follows from Lemma 14 and Proposition 15 since diametral extensions $L_1 \in \mathcal{L}(id)$, $L_2 \in \mathcal{L}(\overline{id})$ have $\mathrm{nr}_{k,k}(L_1, L_2) = 0$ and $\mathrm{nr}_{1,k}(L_1, L_2) = \alpha_n^k$, so they have $V_\mathcal{A} \cup V_\mathcal{B} = V$ for some minimal vertex cover V of $G_n^k(\overline{id})$. Automorphisms of $B_n^{1,k}$ only allow an arbitrary order of atoms in L_1. □

For $k = 2$ and $n \geq 4$ we have additional diametral pairs obtained from an additional minimal vertex-edge cover of $G_n^2(\overline{id})$. Let V be a minimal vertex cover of $G_n^2(\overline{id})$ where $n \geq 4$. We say that $L_1 \in \mathcal{L}_n^2(id)$, $L_2 \in \mathcal{L}_n^2(\overline{id})$ are *V-almost compatible* if for every 2-set S except $2(n-1)$,

a) S is in the ith slot in L_1 and in the last slot in L_2 if $\mathcal{B}_S \subseteq V$,

b) S is in the jth slot in L_2 and in the last slot in L_1 if $\mathcal{A}_S \subseteq V$,

c) $2(n-1)$ is in the $(n-1)$th slot both in L_1 and in L_2,

d) all pairs of 2-sets except $\{2(n-1), 1n\}$ are reversed in L_1, L_2,

where $i = \max(S) = n - |\mathcal{A}_S|$, $j = n - \min(S) + 1 = n - |\mathcal{B}_S|$. Note that for $k = 2$ we have $\mathcal{Z}(L_1, L_2) = \{1n\}$, so c) does not contradict d). Similarly as above, we have the following equivalent definition.

Proposition 18. *Let $n \geq 4$ and V be a minimal vertex cover of $G_n^2(\overline{id})$. Then $L_1 \in \mathcal{L}_n^2(id)$, $L_2 \in \mathcal{L}_n^2(\overline{id})$ are V-almost compatible if and only if $V_\mathcal{A} \cup V_\mathcal{B} = V \setminus V(\mathcal{C}_{2(n-1)})$ and $\{2(n-1), 1n\}$ is the only pair of 2-sets unreversed in L_1, L_2 where*

$$V_\mathcal{A} = \{\{S, u\} \in \mathcal{A};\ u <_{L_1} S\}, \quad V_\mathcal{B} = \{\{S, u\} \in \mathcal{B};\ u <_{L_2} S\}.$$

Proof. Similar to the proof of Proposition 15, omitted. □

An example of V-almost compatible pair for $n = 4$ is $L_1 = (1, 2, 12, 3, 13, 23, 4, 14, 24, 34)$, $L_2 = (4, 3, 34, 2, 24, 23, 1, 14, 13, 12)$. Note that d) implies that $2(n-1)$ is last in the $(n-1)$th slot in both L_1, L_2. Every V-almost compatible pair can be obtained from some V-compatible pair by moving $2(n-1)$ to last positions in the $(n-1)$th slots in both linear extensions.

It is remarkable that in the above example the order in L_1, L_2 is in fact σ-reversed lexicographical and $\overline{\sigma}$-reversed lexicographical, respectively, where $\sigma = id$. However, it turns out that this is possible only for $n \leq 4$.

Theorem 19. *Linear extensions $L_1 \in \mathcal{L}_n^2(id)$, L_2 of $\mathcal{B}_n^{1,2}$ for $n \geq 2$ are diametral if and only if they are V-compatible or V-almost compatible for some minimal vertex cover V of $G_n^2(\overline{id})$. All other diametral linear extensions are obtained by automorphisms of $B_n^{1,2}$.*

Proof. The sufficiency follows from $\mathrm{nr}(L_1, L_2) = \alpha_n^2$ and $L_2 \in \mathcal{L}(\overline{id})$ for every V-compatible or V-almost compatible L_1, L_2. The case of V-compatible L_1, L_2 is the same as in Theorem 17. For V-almost compatible L_1, L_2 we have $\mathrm{nr}_{1,2}(L_1, L_2) = |V_\mathcal{A} \cup V_\mathcal{B}| = \alpha_n^2 - 1$ and $\mathrm{nr}_{2,2}(L_1, L_2) = 1$ by Proposition 18.

The necessity follows from Lemma 14 and Propositions 15 and 18 since diametral extensions $L_1 \in \mathcal{L}(id)$, $L_2 \in \mathcal{L}(\overline{id})$ have $\mathrm{nr}_{2,2}(L_1, L_2) = 0$ and $\mathrm{nr}_{1,2}(L_1, L_2) = \alpha_n^2$, or $\mathrm{nr}_{2,2}(L_1, L_2) = 1$ with unreversed $\{2(n-1), 1n\}$ and $\mathrm{nr}_{1,2}(L_1, L_2) = \alpha_n^2 - 1$. The first case is the same as in Theorem 17. In the latter case, $(V_\mathcal{A} \cup V_\mathcal{B}, \{\{2(n-1), 1\}, \{2(n-1), n\}\})$ is a vertex-edge cover of $G(\overline{id})$ of size α_n^2. It follows that $V_\mathcal{A} \cup V_\mathcal{B} = V \setminus V(\mathcal{C}_{2(n-1)})$ for some minimal vertex cover V of $G_n^2(\overline{id})$. Automorphisms of $B_n^{1,2}$ only allow an arbitrary order of atoms in L_1. □

One may ask whether a given linear extension can be in more than one diametral pair. It turns out from the constructions of V-compatible and V-almost compatible pairs that the answer is negative up to $k = 2$, $n \geq 4$ and a linear extension L_1 that is V-compatible with some L_2 and has $2(n-1)$ on the last position in the $(n-1)$th slot. In this case L_1 is almost V-compatible with L_2' obtained from L_2 by moving $2(n-1)$ to the last position in the $(n-1)$th slot. See the above examples of V-compatible and V-almost compatible pairs.

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