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Prohlašuji, že jsem svou diplomovou práci napsal samostatně a výhradně s použitím citovaných pramenů. Souhlasím se zapůjčováním práce.

V Praze dne Jiří Fink

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Abstrakt:

Jedním ze základních problémů moderní statistické fyziky je snada porozumět frustraci a chaosu. Základním modelem je konečně dimenzionální Edwards-Anderson Ising model. V optimalizaci to odpovídá zkoumání minimálních T-joinů v konečných mřížkách s náhodnými váhami na hranách.

V této práci studujeme "random join", což je náhodná cesta mezi dvěma pevně danými vrcholy. Původní definice je příliš složitá, a tak jsme ukázali jednodušší. Tato definice je použita k přesnému výpočtu "random join" na kružnicí. Také jsme ukázali speciální algoritmus, který hledá cestu v mřížce s danými hranami. Tento algoritmus může být použit k experimentálnímu studování "random join".

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Abstract:

One of the basic streams of modern statistics physics is an effort to understand the frustration and chaos. The basic model to study these phenomena is the finite dimensional Edwards-Anderson Ising model. In discrete optimisation this corresponds to the minimal T-joins in a finite lattice with random weights of edges.

This thesis studies a random join which is a random path between two given vertices. The original definition of the random join is very complex, and we have managed to find an equivalent one which is more natural. We use our definition to exactly compute the random join on circles. We also propose an algorithm which finds the shortest path in a large lattice with given weights of edges. This algorithm can be used for an experimental study of the random join.

Chapter 1

Introduction

1.1 Graph theory

Let M and K be finite sets and n be a natural number. We will use the following five symbols for special sets:

2^M	$= \{K \mid K \subset M\}$	the family of all subsets.
$\binom{M}{n}$	$= \{K \mid K \subset M, \lvert K\rvert = n\}$	the family of subsets of size n;
$\binom{M}{even}$	$= \{K \mid K \subset M, \lvert K\rvert \text{ is even}\}$	the family of subsets of an even size;
$M \times K$	$= \{(x,y) \mid x \in M,\ y \in K\}$	the Cartesian product;
$M \triangle K$	$= (M \setminus K) \cap (K \setminus M)$	the symmetric difference.

We will very often talk about a family of subsets of an even size of a finite set. We write *even subset* instead of *subset of even size* to make notation simple.

A *graph* is an ordered pair of disjoint sets (V, E) such that E is a subset of the set $\binom{V}{2}$ of unordered pairs of V. Unless it is explicitly stated otherwise, we consider only finite graphs, that is, V and E are always finite. The set $V = V(G)$ is the set of *vertices*, and $E = E(G)$ is the set of *edges*. An edge $\{x, y\}$ is said to join the vertices x and y and is denoted by xy. Thus xy and yx mean exactly the same edge; the vertices x and y are the end-vertices of this edge. If $xy \in E(G)$, then x and y are *adjacent* vertices of G, and the vertices x and y are *incident* with the edge xy. Two edges are *adjacent* if they have exactly one common end-vertex. The *degree* of a vertex v is the number of the vertices which are adjacent to v; we denote it by $deg(v)$.

By definition a graph does not contain a *loop*, an edge joining a vertex to itself; neither does it contain *multiple edges*, that is, several edges joining the same two vertices. In a *multigraph* both multiple edges and multiple loops are allowed. A loop is a special edge.

A *weighted graph* (V, E, ω) is a graph (V, E) whose each edge e has a weight $\omega(e)$. It is useful to extend the weight function ω to sets of edges. We define $\omega(J)$ where $J \subset E$ as $\sum_{e \in J} \omega(e)$.

We say that $G' = (V', E')$ is a *subgraph* of $G = (V, E)$ if $V' \subset V$ and $E' \subset E$. In this case we write $G' \subset G$. If G' contains all edges of G that join two vertices in V', then G' is said to be the subgraph *induced* by V' and is denoted by $G[V']$.

A *path of length l* is a graph

$$P_l = (\{v_0, v_1, \dots, v_l\}, \{v_0v_1, v_1v_2, \dots, v_{l-1}v_l\}).$$

This path P_l is usually denoted by $v_0 v_1 \cdots v_l$. The vertices v_0 and v_l are the *end-vertices* of P_l. We say that P_l is a path *from* v_0 to v_l, or an $v_0 - v_l$ path. A *circle of length* $l \geq 3$ is a graph

$$C_l = (\{v_1, v_2, \ldots, v_l\}, \{v_1v_2, v_2v_3, \ldots, v_{l-1}v_l, v_lv_1\}).$$

For simplicity, this circle is denoted by $v_1 v_2 \cdots v_n$. Vertices are pairwise different in the path and in the circle.

A graph is *connected* if for every pair $\{x, y\}$ of distinct vertices there is a path from x to y. A *maximal connected subgraph* is a *component* of the graph. A *cut-vertex* is a vertex whose deletion increases the number of components. Similarly, an edge is a *bridge* if its deletion increases the number of components. A connected graph without any cut-vertex is called 2-connected. A graph without any circles is a *forest*; a *tree* is a connected forest.

Let $G = (V, E)$ be a graph and let T be a subset of an even number of vertices of G. We say that a set J of edges of G is a *T-join* if each vertex v of G is incident with an even number of edges of J if, and only if, $v \notin T$.

1.2 Probability

The Gaussian distribution $\mathcal{N}(\mu, \sigma)$ of mean μ and variance σ^2 is given by density function

$$\phi_{\mu,\sigma^2}(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

We will mostly use *the standard Gaussian distribution* $\mathcal{N}$ of mean $\mu = 0$ and variance $\sigma^2 = 1$ and we denote its density function by ϕ. The *positive Gaussian distribution* $\mathcal{N}^+$ is given by density function

$$\phi^+(x) = \begin{cases} \sqrt{\frac{2}{\pi}}\, e^{-\frac{1}{2}x^2} & \text{if } x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

It is the distribution of absolute value of random variable chosen from the standard Gaussian distribution. Sometimes it is useful to extend those functions for a real vector $x = (x_1, \ldots, x_n)$ and denote $\phi(x) = \prod_{i=1}^n \phi(x_i)$. We can observe that

$$\phi^+(x) = \begin{cases} 2^n\, \phi(x) & \text{if } x \geq 0 \\ 0 & \text{otherwise.} \end{cases} \tag{1.1}$$

1.3 A random join

A *random join* between vertices v_u and v_l in a graph $G = (V, E)$ is chosen in the following way: The weights of the edges of G are independently chosen from the positive Gaussian distribution. A subset of vertices T_1 is chosen uniformly from $\binom{V}{even}$, which denotes the family of even subsets of vertices. The second subset of vertices is $T_2 = T_1 \triangle \{v_u, v_l\}$. Let J_i be the minimal-weight T_i-join, for $i = 1, 2$. The symmetric difference of J_1 and J_2 is the random join.

The main problems addressed in this thesis are the properties of the random joins in a finite 2-dimensional lattice. A *finite 2-dimensional lattice* $C(n, k)$ is a graph which vertices have two coordinates. The horizontal coordinates run through integers from $-k$ to k and the vertical coordinates run through integers from $-n$ to n. Two vertices of the graph are connected by an edge if their distance is 1. If not written otherwise a *lattice* means a finite 2-dimensional lattice $C(n, k)$. Let us denote by v_u and v_l the vertices in the middle of the upper and the lower horizontal border of $C(n, k)$.

If it is not written otherwise, a graph $G = (V, E)$ is a connected graph in this thesis.

Chapter 2

T-join

In this chapter we explain the background for main theorems. We present some condition for a T-join to be minimal, and we talk about unique minimal T-join. We also write some known theorems about the Eulerian graphs and the Gaussian distribution and the Lebesgue integral and measure.

2.1 The Eulerian graph

In this section we study connections between T-joins and the Eulerian set of edges.

Let us recall definition of T-join. Let T be an even subset of vertices of G. We say that a set J of edges is a *T-join* if each vertex v of G is incident with an even number of edges of J if, and only if, $v \notin T$. We have to say first when a T-join exists.

Proposition 1. *Let $G = (V, E)$ be an arbitrary graph and $T \in \binom{V}{even}$. Graph G has a T-join if, and only if, every component of G has even number of vertices in T.*

As we wrote at the end of the first chapter, we consider that graph G is connected. Hence the graph G has a T-join for every set T of even size. Later in this section, we show how many T-joins the graph G has.

We said that T is an even subset of vertices. A natural question is how many even subsets a finite and non-empty set V has. We can find the answer in a very nice book [14] written by Matoušek and Nešetřil.

Proposition 2. *Let M be a finite and non-empty set. Number of even subsets of M is $2^{|M|-1}$.*

We often speak about parity, especially about even numbers. We know that the symmetric difference of two sets A and B has an even size if, and only if, sets A and B have the sizes of the same parity. This fact is used in a proof of the following observation about T-joins.

Proposition 3. *Let $J_1, J_2 \subset E$ and $T_1, T_2 \in \binom{V}{even}$. Let J_1 be a T_1-join of G. Then J_2 is a T_2 of G if, and only if, $J_1 \triangle J_2$ is a $(T_1 \triangle T_2)$-join of G.*

This proposition is proved in Cook's book [4]. It is used in most observations and theorems of this thesis and we use it without explicit reference.

We continue to the Eulerian graph. We say that a graph is *Eulerian* if every vertex has an even degree. A subset J of edges E is Eulerian if a graph (V, J) is Eulerian. It is easy to see that every circle in a graph G is an Eulerian subset. We will use following theorem noticed by Veblen in 1912. Readers can find a proof of the theorem in Bollobas's book [2].

Proposition 4. *The edge set of a graph G can be partitioned into circles if, and only if, the graph G is Eulerian.*

A family of Eulerian subset of given graph G, denoted by $\mathcal{K}_G$, is an interesting vector space which is studied in Matoušek's and Nešetřil's book [14]. We will need a corollary, which says how large the family is.

Proposition 5. *Number of Eulerian subset of graph $G = (V, E)$ which has k components is $2^{|E|-|V|+k}$.*

From this proposition follows that a connected graph G has $2^{|E|-|V|+1}$ Eulerian subsets.

It follows from propositions 2 and 5 that the number of possible sets T multiplied by the number of Eulerian subset is $2^{|E|}$, which is number of all subsets of edges. Is it a coincidence?

Let us fix a $T \in \binom{V}{even}$. By proposition 1 there exists a T-join J. If C is an Eulerian subset, then $J \triangle C$ is still a T-join. And we have $2^{|E|-|V|+1}$ Eulerian subsets by proposition 5. Hence, there exist $2^{|E|-|V|+1}$ T-joins for every $T \in \binom{V}{even}$.

When a set $J \subset E$ is given, we easily find such unique set $T \subset V$ that J is a T-join by counting parity of degree in a graph (V, J). We denote such set T as T_J.

Let us fix a representative T-join J_T for every $T \in \binom{V}{even}$ and consider a set $J \subset E$. We can find such unique $T_J \in \binom{V}{even}$ that J is a T_J-join. Moreover, we have a representative T_J-join J_{T_J} and a unique Eulerian subset $J \triangle J_{T_J}$. Hence there exist a unique $T_J \in \binom{V}{even}$ and an unique Eulerian subset for every subset of edges J.

This discussion proves the following theorem.

Theorem 1. *There is one-to-one correspondence between $\binom{V}{even} \times \mathcal{K}_G$ and 2^E.*

2.2 Weighted T-join

In this section we study the minimal-weight T-join and the weights of edges without a negative circle. We show necessary and sufficient condition for a T-join to be minimal.

We consider a weight functions of edges $\omega : E \to \mathbb{R}$ in this section. We say that the *weight has no negative circle* if there is no Eulerian subset of edges J so that sum of the weights of the edges in J is negative.

Proposition 6. *The weighted graph $G = (V, E, \omega)$ has no negative circle if, and only if, it has no negative Eulerian subset.*

Proof. One implication is trivial because every circle is an Eulerian subset. By proposition 4 an Eulerian graph can be partitioned into disjoin circles and sum of non-negative weights of those circles cannot be a negative Eulerian subset. □

In the introduction we talked about minimal-weight T-join. We will often talk about it so we will for short use term *minimal T-join* instead of *minimal-weight T-join.* Sometime we will work with more weight functions on edges, so we will write that J *is a minimal* T*-join with respect to* ω to emphasise the weight we consider.

It is known that each minimal T-join is a forest when the weights of all the edges are positive. But we work with the weights which have no negative circle so we must be more careful.

Proposition 7. *If the weight function* ω *has no negative circle and* J *is the unique minimal* T*-join for a set* $T \in \binom{V}{even}$*, then a graph* (V, J) *is a forest.*

Proof. For contradiction we suppose that the graph (V, J) contains a circle $C \subset J$. Then

$$\omega(J \triangle C) = \omega(J \setminus C) = \omega(J) - \omega(C).$$

Hence, C is a negative circle or $\omega(J \triangle C) \leq \omega(C)$ where $J \triangle C$ is a T-join, which is a contradiction. □

When we attentively read the proof of the last observation, we observe that there always exists a minimal T-join which is a forest. Hence, when minimal T-join is not unique, we consider an arbitrary minimal T-join, which is a forest.

We will often need to change the sign of the weight of the edges which belong into a minimal T-join. Let us define this formally.

Definition 1. *Let* J *be a subset of edges of a weighted graph* $G = (V, E, \omega)$*. A function* $\omega_J : E \to \mathbb{R}$ *is defined by*

$$\omega_J(e) = \begin{cases} \omega(e) & \text{if } e \notin J \\ -\omega(e) & \text{otherwise} \end{cases}$$

for all edges $e \in E$*. If* J *is a minimal* T*-join with respect to* ω*, we denote* ω_J *by* ω^T*.*[1]

We need a simple, but useful, observation.

Lemma 1. *Let* J *and* K *be arbitrary subsets of* E*. Then* $\omega_J(K \triangle J) = \omega(K) - \omega(J)$*.*

Proof.

$$\begin{aligned} \omega_J(K \triangle J) &= \omega_J((K \setminus J) \cup (J \setminus K)) \\ &= \omega_J(K \setminus J) + \omega_J(J \setminus K) \\ &= \omega(K \setminus J) - \omega(J \setminus K) \\ &= (\omega(K \setminus J) + \omega(K \cap J)) - (\omega(J \setminus K) + \omega(K \cap J)) \\ &= \omega(K) - \omega(J). \end{aligned}$$

□

When ω and J satisfy the conditions of proposition 7, then it is not surprising that ω_J has no negative circle. But the condition to ω is not necessary! Moreover we can formulate the proposition as an equivalence.

[1] If a minimal T-join is not unique, we consider an arbitrary minimal T-join which is a forest.

Theorem 2. *Let J be a T-join where $T \in \binom{V}{even}$. Then J is a minimal T-join if, and only if, ω_J has no negative circle.*

Proof. If ω_J contains a negative circle C, then

$$0 > \omega_J(C) = \omega_J((J \triangle C) \triangle J) = \omega(J \triangle C) - \omega(J).$$

Hence $\omega(J \triangle C) < \omega(J)$ and J is not a minimal T-join.

If J is not a minimal T-join, then there exists a T-join J' of smaller weight. Then

$$\omega_J(J \triangle J') = \omega(J') - \omega(J) < 0.$$

Since $J \triangle J'$ is a Eulerian subset, ω_J must contains a negative circle by proposition 6. □

2.3 Unicity of a minimal T-join

As we discuss below proposition 7, a minimal T-join does not need no be unique for every set T and every weight function. In this section we present a sufficient condition for the distribution function of the weight function so that ambiguous minimal T-joins do not occur too often.

We denote it by $X \sim \mathcal{U}[M]$, a random variable X *uniformly* chosen from a finite set M.

Lemma 2. *If $J \sim \mathcal{U}\left[2^E\right]$, then $T_J \sim \mathcal{U}\left[\binom{V}{even}\right]$.*

Proof. Let us consider a fixed $T \in \binom{V}{even}$. By theorem 1 there exist such $2^{|E|-|V|+1}$ subsets $J \subset E$ that J is a T-join. Every $J \subset E$ has probability $\frac{1}{2^{|E|}}$ to be chosen. Hence the probability, that J is a T-join where $J \sim \mathcal{U}\left[2^E\right]$, is $\frac{1}{2^{|V|-1}}$. □

The last lemma is not exactly what we need. We are studying T-join having no negative circle. We would like to show that if $J \sim \mathcal{U}\left[2^E\right]$ has no negative circle, then $T_J \sim \mathcal{U}\left[\binom{V}{even}\right]$. It holds under the condition that the considered weight function ω has a unique minimal T-join for all $T \in \binom{V}{even}$ because there exists one set $J \subset E$, such that J has no negative circle and J is a T-join. But we have not a fixed weight but a random one in the random join. Hence, we need to show that we have not two or more minimal T-joins too often.

Let us denote by $\mathbb{R}^E$ the $|E|$ dimensional real space in which the coordinates are labelled by edges of the graph G. Similarly, we denote by $\mathbb{N}^E$ the set of integer vectors of length $|E|$.

Definition 2. *We say that a distribution function of a weight function ω is unique if the probability, then there exists a nonzero integer linear combination $n \in \mathbb{N}^E$ satisfying $\sum_{e \in E} n_e \omega(e) = 0$ is zero.*

We will later prove that the Gaussian distribution is unique.

Proposition 8. *If $T \in \binom{V}{even}$ and distribution function of a weight function ω is unique than a minimal T-join is unique with probability one.*

Proof. Let us consider two different T-joins J_1 and J_2 having the same weight, i.e. $\omega(J_1) = \omega(J_2)$. We define two integer linear combinations k and l by

$$k_e = \begin{cases} 1 & \text{if } e \in J_1 \\ 0 & \text{otherwise} \end{cases}$$

for all $e \in E$ and l, is a similar combination for J_2. We define n as difference of k and l, i.e. $n_e = k_e - l_e$ for all $e \in E$. We compute that

$$\sum_{e \in E} n_e \omega(e) = \sum_{e \in E} k_e \omega(e) - \sum_{e \in E} l_e \omega(e) = \omega(J_1) - \omega(J_2) = 0$$

but it happens with probability zero. Hence we proved that there exist two T-joins having the same weight with probability zero. $\square$

2.4 Mathematical analysis

We will also need some terminology and theorems from mathematical analysis but we will not explain it in such detail. We only say theorems which are directly used is this thesis. Theorems from this section were found in the book by Billingsley [16] and in our lecture notes.

In this thesis we use only the Lebesgue integral and the Lebesgue measure. We denote measure of a set M by $\lambda(M)$. We write $f \in L(M)$ where $f : M \to \mathbb{R}$ if the set $M \subset \mathbb{R}^n$ is measurable and the Lebesgue integral $\int_M f$ exists. There are two well-known theorems.

Theorem 3 (Fubini). *If $M \subset \mathbb{R}^{m+k}$ and $f \in L(M)$, then*

- *function $F(x) = \int_{M_x} f(x, y) \, \mathrm{d}y$ is defined for almost every $x \in \mathbb{R}^m$*
- *$F(x) \in L(\mathbb{R}^m)$*
- *$\int_M f = \int_{\mathbb{R}^m} F(x) \, \mathrm{d}x$*

where $M_x = \{ y \in \mathbb{R}^k \mid (x, y) \in M \}$.

Theorem 4 (substitution). *Let function $\psi : U \to \mathbb{R}^n$ be a regular bijection where $U \subset \mathbb{R}^n$ is an open set. Let $B \subset \psi(U)$ be a measurable set. If Lebesgue integral $\int_B f$ exists, then an equation*

$$\int_B f(y) \, \mathrm{d}y = \int_{\psi^{-1}(B)} f(\psi(x)) \left| det J_{\psi(x)} \right| \mathrm{d}x$$

holds.

A symbol $J_{\psi(x)}$ means the Jacobian matrix. We will mostly use a two-dimensional ellipse substitution $\psi(r, \alpha) = (ar \cos(\alpha), br \sin(\alpha))$ where a and b are positive real numbers. A determinant of the Jacobian matrix of this substitution is abr.

As we discuss in proposition 8, we will work with probability zero. Hence, we will need some theorems about sets of measure zero.

Proposition 9. *Every countable subset of real numbers has measure zero. Every countable union of sets of measure zero has again measure zero.*

Proposition 10. *Let us consider a function $f \in L(M)$. If $\lambda(M) = 0$, then $\int_M f = 0$. Moreover if f is a positive function on the set M and $\int_M f = 0$, then $\lambda(M) = 0$.*

2.5 The Gaussian distribution

In this section we prove that the Gaussian distribution is unique and we present a theorem about a combination of the Gaussian and the uniform distribution.

The following interesting and important observation about the Gaussian distribution is proven in book [15].

Proposition 11. *The sum of two random variables chosen independently from the Gaussian distributions with means μ_1 and μ_2 and variances σ_1^2 and σ_2^2 is again a random variable chosen from the Gaussian distribution with mean $\mu_1 + \mu_2$ and variance $\sigma_1^2 + \sigma_2^2$.*

From that observation it is easy to prove by induction another useful fact.

Proposition 12. *The sum of n random variables chosen (mutually) independently from the standard Gaussian distribution is a random variable chosen from the Gaussian distribution with zero mean and variance n.*

It is not difficult to prove a general version of those propositions.

Proposition 13. *A nonzero integer linear combination of random variables chosen (mutually) independently from the Gaussian distribution is again a random variable chosen from the Gaussian distribution.*

We denote by $\omega \sim \mathcal{N}_G$ and $\omega \sim \mathcal{N}_G^+$ random weight function ω where $\omega(e)$ are independently chosen from the standard Gaussian distribution and the positive Gaussian distribution for all $e \in E(G)$, respectively.

In previous sections we discussed minimal T-joins which are not unique. We said that a distribution function of a weight function ω is unique if the probability that there exists a nonzero integer linear combination $n \in \mathbb{N}^E$ satisfying $\sum_{e \in E} n_e \omega(e) = 0$ is zero. Then we proved in proposition 8 that minimal T-join is unique with probability one if $T \in \binom{V}{even}$ and the distribution function of a weight function ω is unique. Now we prove that the Gaussian distribution is unique.

Theorem 5. *The Gaussian distribution of a weight function ω is unique.*

Proof. We start by a simple observation: probability that random variable X chosen from the Gaussian distribution is equal to zero is zero. Measure of a set containing only number zero is zero by proposition 9 and $\mathrm{P}(X = 0) = \int_{\{0\}} \Phi_{\mu,\sigma^2} = 0$ by proposition 10.

Now we prove that theorem holds for a fix nonzero integer linear combination, i.e. probability that $\sum_{e \in E} n_e \omega(e) = 0$ is zero where $n \in \mathbb{N}^E$ is a fix nonzero integer linear combination and weights ω are chosen from the Gaussian distribution. By proposition 13 the sum is a random variable X chosen from the Gaussian distribution. Then by previous paragraph

$$\mathrm{P}\left(\sum_{e \in E} n_e \omega(e) = 0\right) = \mathrm{P}(X = 0) = 0.$$

Using proposition 10 we know that measure of a set $\left\{\omega \in \mathbb{R}^E \;\middle|\; \sum_{e \in E} n_e \omega(e) = 0\right\}$ is zero. There exist only countably many integer linear combinations so a union over all nonzero integer linear combinations of those sets has measure zero by proposition 9. Hence probability that there exists a nonzero integer linear combination satisfying the formula is an integral over a set of zero measure which is equal to zero by proposition 10. □

We will often combine the Gaussian and the uniform distribution. An example of this combination is following observation from [5].

Proposition 14. *If $U \sim \mathcal{U}[\{\pm 1\}]$ and $X \sim \mathcal{N}$, then $XU \sim \mathcal{N}$.*
If $U \sim \mathcal{U}[\{\pm 1\}]$ and $X \sim \mathcal{N}^+$, then $XU \sim \mathcal{N}$.

We will sometimes have a sign U_e and weight $\omega(e)$ for every edge e where signs are uniformly chosen from $\{\pm 1\}$ and weight function is chosen from $\mathcal{N}_G^+$. Those random variables are mutually independent. We denote a vector labelled by edges which has value $U_e\omega(e)$ on index e by $U\omega$. We would like to compute the probability that the product $U\omega$ has some property, e.g. it belongs into a measurable set $M \subset \mathbb{R}^E$. Then we show that this probability is equal to $\mathrm{P}(x \in M)$ where x is chosen from $\mathcal{N}$. Hence, distribution functions of $U\omega$ and x are the same.

Before we start any integration, we show what is our probability space and the distribution function. The random vector (U, ω) belongs into space $\{\pm 1\}^E \times \mathbb{R}^E$. The distribution function $d : \{\pm 1\}^E \times \mathbb{R}^E \to \mathbb{R}$ of (U, ω) is $d(u, \omega) = \frac{1}{2^{|E|}}\Phi^+(\omega)$. And we can start the integration.

$$\mathrm{P}(U\omega \in M) = \sum_{U \in \{\pm 1\}^E} \int_{U\omega \in M} d(u, \omega)\,\mathrm{d}\omega = \sum_{U \in \{\pm 1\}^E} \int_{\substack{\omega \geq 0 \\ U\omega \in M}} \Phi(\omega)\,\mathrm{d}\omega$$

We used equation 1.1 in the last step. Let us express probability $\mathrm{P}(x \in M)$.

$$\mathrm{P}(x \in M) = \int_{x \in M} \Phi(x)\,\mathrm{d}x$$

Now, we need to prove that last two expressions are equal. Instead of proving exactly this equality we prove somewhat general version.

We say that a function $f : \mathbb{R}^n \to \mathbb{R}$ is even if for every $x \in \mathbb{R}^n$ and every $u \in \{\pm 1\}^n$ hold $f(ux) = f(x)$.

Lemma 3. *Let n be a natural number. Let $M \subset \mathbb{R}^n$ be a measurable set and $f \in L(M)$ be a function. Then*

$$\int_{x \in M} f(x)\,\mathrm{d}x = \sum_{U \in \{\pm 1\}^n} \int_{\substack{\omega \geq 0 \\ U\omega \in M}} f(U\omega)\,\mathrm{d}\omega.$$

Moreover if the function is even, then

$$\int_{x \in M} f(x)\,\mathrm{d}x = \sum_{U \in \{\pm 1\}^n} \int_{\substack{\omega \geq 0 \\ U\omega \in M}} f(\omega)\,\mathrm{d}\omega.$$

Proof. Using substitution $y = U\omega$, we reach

$$\sum_{U \in \{\pm 1\}^n} \int_{\substack{\omega \geq 0 \\ U\omega \in M}} f(U\omega)\,\mathrm{d}\omega = \sum_{U \in \{\pm 1\}^n} \int_{\substack{Uy \geq 0 \\ y \in M}} f(y)\,\mathrm{d}y.$$

We need to prove

$$\int_{x \in M} f(x)\,\mathrm{d}x = \sum_{U \in \{\pm 1\}^n} \int_{\substack{Uy \geq 0 \\ y \in M}} f(y)\,\mathrm{d}y.$$

If we carefully view the internal integral in the sum, we can see that it integrates over intersection of one octant and the set M. The sum summarises all octane so the left side integrate of whole M.

If the function is even, then $f(U\omega) = f(\omega)$ for all $U \in \{\pm 1\}^n$ and $\omega \in {\mathbb{R}^+}^n$. $\square$

This discussion proves the following theorem:

Theorem 6. *Let* $U \sim \mathcal{U}\left[\{\pm 1\}^E\right]$ *and* $\omega \sim \mathcal{N}_G^+$ *and* $x \sim \mathcal{N}_G$*. Then* $\mathrm{P}\left(U\omega \in M\right) = \mathrm{P}\left(x \in M\right)$ *for all measurable set* $M \subset \mathbb{R}^E$*.*

Chapter 3

Random join

In this chapter we study the random join on a connected graph $G = (V, E)$. We prove that the random join between vertices v_u and v_l is a $v_u - v_l$ path. We present simpler way to generate the random join and we compute probabilities of both random joins on circles.

3.1 A path

In this section we prove a theorem which shows us that it may possible to simplify the original definition of the random join. We use it to prove that the random join is a path. We will discuss a significance of the theorem in next section.

The important theorem is:

Theorem 7. *Let $T_1, T_2 \in \binom{V}{even}$ and $\omega : E \to \mathbb{R}$ be a weight function and J_1 and J_2 be arbitrary T_1-join and T_2-join, respectively. Then J_2 is a minimal T_2-join with respect to ω if, and only if, $J_1 \triangle J_2$ is a minimal $(T_1 \triangle T_2)$-join with respect to ω_{J_1}.*

Proof. Let us start the proof by the left-to-right implication. For contradiction we suppose that $J_1 \triangle J_2$ is not a minimal $(T_1 \triangle T_2)$-join with respect to ω_{J_1} so there exists a $(T_1 \triangle T_2)$-join J such that $\omega_{J_1}(J) < \omega_{J_1}(J_1 \triangle J_2)$. Using lemma 1 we have

$$\begin{aligned}
\omega_{J_1}(J) &< \omega_{J_1}(J_1 \triangle J_2) \\
\omega_{J_1}((J \triangle J_1) \triangle J_1) &< \omega_{J_1}(J_2 \triangle J_1) \\
\omega(J \triangle J_1) - \omega(J_1) &< \omega(J_2) - \omega(J_1) \\
\omega(J \triangle J_1) &< \omega(J_2)
\end{aligned}$$

where $J \triangle J_1$ is a T_2-join. It is a contradiction to minimality of the T_2-join J_2.

The other implication is also proven by contradiction. We suppose that J_2 is not a minimal T_2-join with respect to ω. Then there exists a T_2-join J of a smaller weight. We use lemma 1 again

$$\begin{aligned}
\omega(J) &< \omega(J_2) \\
\omega(J) - \omega(J_1) &< \omega(J_2) - \omega(J_1) \\
\omega_{J_1}(J \triangle J_1) &< \omega_{J_1}(J_2 \triangle J_1),
\end{aligned}$$

but $J \triangle J_1$ is a $(T_1 \triangle T_2)$-join. $\square$

From this theorem it follows that a random join is already a $v_u - v_l$ path. Let us introduce notation first. A set $T_1 \in \binom{V}{even}$ and a non-negative weight function ω and vertices v_u and v_l are given. We denote $T_1 \triangle \{v_u, v_l\}$ by T_2 and a minimal T_i-join by J_i, for $i = 1, 2$. The random join between vertices v_u and v_l is $J_1 \triangle J_2$.

Corollary 1. *The random join between vertices v_u and v_l is a $v_u - v_l$ path for all possible T_1 and ω.*

Proof. By theorem 2 we know that ω_{J_1} has no negative circle since J_1 is a minimal T_1-join. By the theorem 7 the random join $J_1 \triangle J_2$ is a minimal $(T_1 \triangle T_2)$-join with respect to ω_{J_1} so the random join is a minimal $\{v_u, v_l\}$-join with respect to ω_{J_1}. By proposition 7 the random join is a forest. A forest having exactly two vertices of odd degree must be a path. □

We should write that there exists a minimal $(T_1 \triangle T_2)$-join which is a forest as we discuss bellow the proof of proposition 7. But the probability that a minimal T_1-join or a minimal $(T_1 \triangle T_2)$-join are not unique is zero by proposition 8.

3.2 The distribution

In this section we show a simpler way to generate the random join in theorem 8, which is the most important theorem in this thesis.

Let us consider a weight function ω and a T_1-join J_1. If we find the shortest $v_u - v_l$ path P with respect to ω_{J_1}, then P is a minimal $\{v_u, v_l\}$-join with respect to ω_{J_1}. Let us denote $J_2 = J_1 \triangle P$ and $T_2 = T_1 \triangle \{v_u, v_l\}$. We can see that $J_1 \triangle J_2$ is a minimal $(T_1 \triangle T_2)$-join with respect to ω_{J_1}, so J_2 is a minimal T_2-join with respect to ω by theorem 7. Moreover, if J_1 is a minimal T_2-join, then P is a random join.

In short, if we find a minimal T_1-join J_1 with respect to ω and the shortest $v_u - v_l$ path P with respect to ω_{J_1}, then P is a random join. Hence, if we know the weight function ω_{J_1} we do not need random variables T and ω to find the random join! It is possible to generate the random join as the shortest path with respect to weight function ω_{J_1}. But what is the distribution function of the weight function ω_{J_1}?

Let us denote by $\mathcal{N}_G^T$ a distribution function of the weight function ω_{J_1} which is a minimal T_1-join with respect to ω, where a distribution function of ω is chosen from $\mathcal{N}_G^+$, and T_1 is uniformly chosen from $\binom{V}{even}$.

The distribution function $\mathcal{N}_G^T$ is only a formal definition of distribution function ω_{J_1}. This original way for generating the random join is too complex and we tried to find a simpler one. We asked a question: Is it sufficient to find a distribution function of weights of edges for every edge in the graph G, and choose the weights independently likewise at the distribution $\mathcal{N}_G^+$, or do we need one distribution function for whole graph?

A weight of each edge can be negative in the distribution $\mathcal{N}_G^T$, so independently chosen weights can contains a negative circle. Hence, we need a distribution function which allows negative weights and forbids negative circles. A direct option is a choice weight function from $\mathcal{N}_G$ but omits selections having a negative circle. At first we checked that this option generates the same weight functions as the original distribution $\mathcal{N}_G^T$.

Too often we will need a statement "weight function without a negative circle" in mathematical formulas. Let us define a predicate $\mathrm{NNC}(\omega)$ which is true if, and only if, a weighted graph (V, E, ω) has no negative circle.

Proposition 15. *Let us consider*

$$\Omega_1 = \left\{ \omega^T \;\middle|\; \omega : E \to \mathbb{R}_0^+,\, T \in \binom{V}{even} \right\}$$

and

$$\Omega_2 = \{ \upsilon : E \to \mathbb{R} \mid \mathrm{NNC}\,(\upsilon) \}\,.$$

Then $\Omega_1 = \Omega_2$.

Proof. $\Omega_1 \subset \Omega_2$ follows from theorem 2. Let us consider $\upsilon \in \Omega_2$. Let us denote $J = \{e \in E \mid \upsilon(e) < 0\}$ and $T_J = \left\{ v \in V \;\middle|\; deg_{(V,J)}(v) \text{ is odd} \right\}$ and $\omega(e) = |\upsilon(e)|$ for all $e \in E$. It is obvious that J is a T_J-join and $\omega_J(e) = \upsilon(e)$ for all $e \in E$.

We prove that J is a minimal T_J-join. For contradiction, let us suppose that J' is a T_J-join such that $\omega(J') < \omega(J)$. Using lemma 1 we have

$$\upsilon(J \bigtriangleup J') = \omega_J(J \bigtriangleup J') = \omega(J') - \omega(J) < 0.$$

But $J \bigtriangleup J'$ is an Eulerian subset and υ does not contain a negative circle. $\square$

We see that our option generates the same weight functions as the original distribution $\mathcal{N}_G^T$. But we do not know whether they have the same distribution functions. We are going to prove that they do.

We need to define formally a distribution function where weights of edges are chosen from the standard Gaussian distribution but selections having a negative circle are omitted.

Definition 3. *Let* $G = (V, E, \omega)$ *be a weighted connected graph where* $\omega \sim \mathcal{N}_G$. *Let us denote by* $\mathrm{P_{NNC}}\,(G)$ *the probability that a weighted graph* (V, E, ω) *has no negative circle. We define a distribution function*

$$\Phi^G(\omega) = \begin{cases} \frac{\Phi(\omega)}{\mathrm{P_{NNC}}(G)} & \textit{if } \mathrm{NNC}\,(\omega) \\ 0 & \textit{otherwise} \end{cases}$$

for $\omega \in \mathbb{R}^E$. *We denote a probability space* $(\mathbb{R}^E, \Phi^G)$ *by* $\mathcal{N}_G^{NNC}$.

We need to compute a value of $\mathrm{P_{NNC}}\,(G)$, which we will use later.

Proposition 16.

$$\mathrm{P_{NNC}}\,(G) = \frac{1}{2^{|E|-|V|+1}}$$

Proof. Let us denote a set $M = \left\{\omega \in \mathbb{R}^E \mid \mathrm{NNC}(\omega))\right\}$.

$$
\begin{aligned}
\mathrm{P}_{\mathrm{NNC}}(G) &= \int_{x \in M} \Phi(x)\,\mathrm{d}x \\
\text{by lemma 3} & \\
&= \sum_{U \in \{\pm 1\}^n} \int_{\substack{\omega \geq 0 \\ U\omega \in M}} \Phi(\omega)\,\mathrm{d}\omega \\
\text{by equation 1.1} & \\
&= \int_{\omega \geq 0} \sum_{\substack{U \in \{\pm 1\}^n \\ U\omega \in M}} \frac{1}{2^{|E|}} \Phi^+(\omega)\,\mathrm{d}\omega \\
&= \frac{1}{2^{|E|}} \int_{\omega \geq 0} \Phi^+(\omega) \left| \left\{ U \in \{\pm 1\}^E \mid \mathrm{NNC}(U\omega) \right\} \right| \mathrm{d}\omega
\end{aligned}
$$

We have to compute a size of $\left\{ U \in \{\pm 1\}^E \mid \mathrm{NNC}(U\omega) \right\}$. By definition of ω_J the size is equal to a size of $\{J \subset E \mid \mathrm{NNC}(\omega_J)\}$. We know that a minimal T-join is unique with the probability one for all $T \in \binom{V}{even}$ by proposition 8, and the size is equal to a number of minimal T-joins by theorem 2. Finally, using proposition 2, we have

$$
\mathrm{P}_{\mathrm{NNC}}(G) = \frac{1}{2^{|E|}} \int_{\omega \geq 0} 2^{|V|-1} \Phi^+(\omega)\,\mathrm{d}\omega = \frac{1}{2^{|E|-|V|+1}}.
$$

□

Theorem 8. *Probability spaces $\mathcal{N}_G^T$ and $\mathcal{N}_G^{NNC}$ have the same distribution functions for every connected graph G.*

Proof. At first we found a proof of the theorem only for circles. Later we discovered other observations, which enable us to prove this theorem generally. We would like to show both proofs. We show a proof for circles at first then a general proof which does not use the special proof for circles.

We start by proving the theorem for a circle C_n of length n. Let $e_1 \dots e_n$ be edges of the circle C_n. Let us consider a fixed edge $e \in E(C_n)$ and $\omega^T \sim \mathcal{N}_{C_n}^T$ and $\omega_{NNC} \sim \mathcal{N}_{C_n}^{NNC}$. The proof for circles does not show that distribution functions of ω^T and ω_{NNC} are the same, but it only shows that distribution functions of $\omega^T(e)$ and $\omega_{NNC}(e)$ are the same. Distribution functions of weights of edges in the circle C_n are the same because of symmetry. Hence we will compute distribution functions only for an edge e_1.

Let us denote by Z a random variable $\omega^T(e_1)$ where $\omega^T \sim \mathcal{N}_{C_n}^T$. Probability that Z is less than α is $\mathrm{P}(Z < \alpha)$ by definition where α is an arbitrary real number. Let us consider a random vector $(A_1, \dots, A_n)$ chosen from $\mathcal{N}_{C_n}^{NNC}$. What is the probability that A_1 is less than α? It is the conditional probability that $A_1 < \alpha$ in the condition that there is no negative circle. In our case we have only one circle C_n so the condition is $\sum_{i=1}^n A_i > 0$. Therefore the probability that A_1 is less than α is $\mathrm{P}(A_1 < \alpha \mid \sum_{i=1}^n A_i > 0)$.[1] We need to prove that for all real number α holds $\mathrm{P}(Z < \alpha) = \mathrm{P}(A_1 < \alpha \mid \sum_{i=1}^n A_i > 0)$.

At first, we express the random variable Z using a set T chosen uniformly from $\binom{V}{even}$ and a weight function chosen from $\mathcal{N}_{C_n}^+$. Let us consider a fix $T \in \binom{V}{even}$. Edges of the

[1]We should write $\sum_{i=1}^n A_i \geq 0$, but omission the equality do not change the probability.

circle C_n are partitioned into two disjoint T-joins J_1 and J_2. Without lost of generality we can suppose that J_1 contains the edge e_1. So we can easily find a minimal T-join for given weight function. We consider a sign for every edge

$$U_i = \begin{cases} +1 & \text{if } e_i \in J_1 \\ -1 & \text{otherwise.} \end{cases}$$

We can conversely find T for given $U_1, \ldots, U_n$: a vertex $v_i \in V(C_n)$, which is adjacent to edges e_i and e_{i+1}, belongs into the set T if, and only if, $U_i \neq U_{i+1}$[2]. Moreover when $U_1 = +1$ and we choose $U_i \sim \mathcal{U}\left[\{\pm 1\}\right]$, for $i = 2, \ldots, n$, independently, then appropriate T is chosen from $\mathcal{U}\left[\binom{V}{even}\right]$. So we consider $U_1, \ldots, U_n$ instead of T.

Let $X_1, \ldots, X_n$ be random variables chosen independently from the positive Gaussian distribution. Now we express the random variable Z using $X_1, \ldots, X_n$ and $U_1, \ldots, U_n$. From previous paragraph we know that we have exactly two possible T-joins J_1 and J_2 of weights

$$\sum_{\substack{i=1,\ldots,n \\ U_i=1}} X_i \text{ and } \sum_{\substack{i=1,\ldots,n \\ U_i=-1}} X_i,$$

respectively. Hence J_1 is the minimal T-join if, and only if, $\sum_{i=1}^n U_i X_i < 0$. Because $e_1 \in J_1$ we express the random variable Z as

$$Z = \begin{cases} -X_1 & \text{if } \sum_{i=1}^n U_i X_i < 0 \\ +X_1 & \text{otherwise.} \end{cases}$$

Now we compute the probability $\mathrm{P}\left(Z < \alpha\right)$ using $U_1, \ldots, U_n$ and $X_1, \ldots, X_n$. What is a probability space and a distribution function of those random variables? We know that $U_1 = 1$ and $U_2, \ldots, U_n \sim \mathcal{U}\left[\{\pm 1\}\right]$ and $X_1, \ldots, X_n \sim \mathcal{N}^+$. We can omit the random variable U_1 because it is a fixed number. Hence the probability space is $\{\pm 1\}^{n-1} \times \mathbb{R}^n$. Random variables are mutually independent so the distribution function which we denote by f is

$$f(u_2, \ldots, u_n, x_1, \ldots, x_n) = \frac{1}{2^{n-1}} \prod_{i=1}^n \phi^+(x_i).$$

We denote by x a vector $(x_1, \ldots, x_n)$ and by u a vector $(u_2, \ldots, u_n)$ to make notation simple. We can shortly write $f(u, x) = \frac{1}{2^{n-1}} \phi^+(x)$.

Let us consider that α is a negative real number. The random variable Z is less than

[2] A vertex v_n is adjacent to edges e_n and e_1. Likewise U_{i+1} for $i = n$ means U_1.

α when $x_1 > -\alpha$ and $\sum_{i=1}^n u_i x_i < 0$. Hence the probability $\mathrm{P}(Z < \alpha)$ is

$$\begin{aligned}\mathrm{P}(Z<\alpha) &= \sum_{u\in\{\pm1\}^{n-1}} \int_{\substack{x_1>-\alpha\\ \sum_{i=1}^n u_i x_i<0}} f(u,x)\,\mathrm{d}x \\ &= \sum_{u\in\{\pm1\}^{n-1}} \int_{\substack{x_1>-\alpha,\, x>0\\ \sum_{i=1}^n u_i x_i<0}} \frac{1}{2^{n-1}}\phi^+(x)\,\mathrm{d}x\end{aligned}$$

using equation 1.1 we have

$$\begin{aligned}&= 2\sum_{u\in\{\pm1\}^{n-1}} \int_{\substack{x_1>-\alpha,\, x>0\\ \sum_{i=1}^n u_i x_i<0}} \phi(x)\,\mathrm{d}x \\ &= 2\sum_{u\in\{\pm1\}^{n-1}} \int_{\substack{x_1<\alpha\\ x_2,\dots,x_n>0\\ \sum_{i=1}^n u_i x_i>0}} \phi(x)\,\mathrm{d}x.\end{aligned}$$

We substituted x_1 by $-x_1$ and u_i by $-u_i$, for $i = 2,\dots,n$.

Let us consider that α is a non-negative real number. The random variable Z is between zero and α when $x_1 < \alpha$ and $\sum_{i=1}^n u_i x_i > 0$.

$$\mathrm{P}(0<Z<\alpha) = \sum_{u\in\{\pm1\}^{n-1}} \int_{\substack{x_1<\alpha\\ \sum_{i=1}^n u_i x_i>0}} f(u,x)\,\mathrm{d}x$$

We are using similar operations.

$$= 2\sum_{u\in\{\pm1\}^{n-1}} \int_{\substack{0<x_1<\alpha\\ x_2,\dots,x_n>0\\ \sum_{i=1}^n u_i x_i>0}} \phi(x)\,\mathrm{d}x$$

We can compute the probability $\mathrm{P}(Z < \alpha)$ for the non-negative number α by summing $\mathrm{P}(Z < 0)$ and $\mathrm{P}(0 < Z < \alpha)$[3]:

$$\mathrm{P}(Z<\alpha) = 2\sum_{u\in\{\pm1\}^{n-1}} \int_{\substack{x_1<\alpha\\ x_2,\dots,x_n>0\\ \sum_{i=1}^n u_i x_i>0}} \phi(x)\,\mathrm{d}x.$$

So we have the same probability in both cases, i.e. for $\alpha \in \mathbb{R}$ holds

$$\mathrm{P}(Z<\alpha) = 2\sum_{u\in\{\pm1\}^{n-1}} \int_{\substack{x_1<\alpha\\ x_2,\dots,x_n>0\\ \sum_{i=1}^n u_i x_i>0}} \phi(x)\,\mathrm{d}x.$$

We remember that we must prove the equality $\mathrm{P}(Z < \alpha) = \mathrm{P}(A_1 < \alpha \mid \sum_{i=1}^n A_i > 0)$, where $A_1,\dots,A_n \sim \mathcal{N}$. We expressed the left side of the equality. Now, it is time for the right one. By proposition 12 we known that $\sum_{i=1}^n A_i \sim \mathcal{N}(0,n)$ which imply that $\mathrm{P}(\sum_{i=1}^n A_i > 0) = \frac{1}{2}$. We have

$$\begin{aligned}\mathrm{P}\left(A_1<\alpha \,\middle|\, \sum_{i=1}^n A_i>0\right) &= 2\,\mathrm{P}\left(A_1<\alpha \,\&\, \sum_{i=1}^n A_i>0\right) \\ &= 2\int_{\substack{a_1<\alpha\\ \sum_{i=1}^n a_i>0}} \phi(a)\,\mathrm{d}a.\end{aligned}$$

[3]The probability that Z is equal to zero is zero.

When we prove that an equation

$$\sum_{u\in\{\pm1\}^{n-1}} \int_{\substack{x_1<\alpha \\ x_2,\dots,x_n>0 \\ \sum_{i=1}^n u_i x_i>\beta}} \phi(x)\,\mathrm{d}x = \int_{\substack{a_1<\alpha \\ \sum_{i=1}^n a_i>\beta}} \phi(a)\,\mathrm{d}a$$

holds for all natural number n and real numbers α and β, then from the case $\beta = 0$ the theorem follows for all circles C_n. We prove the last equation by induction on n. For $n = 1$ the equation has no use for our problem but it obviously holds.

We make an induction step from n to $n+1$.

$$\sum_{u_2,\dots,u_{n+1}\in\{\pm1\}} \int_{\substack{x_1<\alpha \\ x_2,\dots,x_{n+1}>0 \\ \sum_{i=1}^{n+1} u_i x_i>\beta}} \phi(x)\,\mathrm{d}x$$

by Fubini theorem

$$= \sum_{u_{n+1}\in\{\pm1\}} \int_{x_{n+1}>0} \phi(x_{n+1}) \sum_{u_2,\dots,u_n\in\{\pm1\}} \int_{\substack{x_1<\alpha \\ x_2,\dots,x_n>0 \\ \sum_{i=1}^n u_i x_i> \\ \beta-u_{n+1}x_{x+1}}} \phi(x_1,\dots,x_n)\,\mathrm{d}x_1\dots x_n\,\mathrm{d}x_{n+1}$$

using induction

$$= \sum_{u_{n+1}\in\{\pm1\}} \int_{x_{n+1}>0} \phi(x_{n+1}) \int_{\substack{a_1<\alpha \\ \sum_{i=1}^n a_i>\beta-u_{n+1}x_{x+1}}} \phi(a_1,\dots,a_n)\,\mathrm{d}a_1\dots a_n\,\mathrm{d}x_{n+1}$$

again Fubini theorem

$$= \sum_{u_{n+1}\in\{\pm1\}} \int_{\substack{x_{n+1}>0 \\ a_1<\alpha \\ \sum_{i=1}^n a_i>\beta-u_{n+1}x_{x+1}}} \phi(x_{n+1})\phi(a_1,\dots,a_n)\,\mathrm{d}a_1\dots a_n,\, x_{n+1}$$

we substitite x_{n+1} by $u_{n+1}a_{n+1}$

$$= \sum_{u_{n+1}\in\{\pm1\}} \int_{\substack{u_{n+1}a_{n+1}>0 \\ a_1<\alpha \\ \sum_{i=1}^n a_i>\beta-a_{n+1}}} \phi(a)\,\mathrm{d}a$$

$$= \int_{\substack{a_{n+1}>0 \\ a_1<\alpha \\ \sum_{i=1}^n a_i+a_{n+1}>\beta}} \phi(a)\,\mathrm{d}a + \int_{\substack{a_{n+1}<0 \\ a_1<\alpha \\ \sum_{i=1}^n a_i+a_{n+1}>\beta}} \phi(a)\,\mathrm{d}a$$

$$= \int_{\substack{a_1<\alpha \\ \sum_{i=1}^{n+1} a_i>\beta}} \phi(a)\,\mathrm{d}a$$

Let us prove the theorem for a connected graph $G = (V, E)$. Let us consider a random set $T \sim \mathcal{U}\left[\binom{V}{even}\right]$ and random weight functions $\omega \sim \mathcal{N}_G^+$ and $Z \sim \mathcal{N}_G^{NNC}$. It is sufficient to prove that an equality $\mathrm{P}\left(\omega^T \in M\right) = \mathrm{P}\left(Z \in M\right)$ holds for every measurable set $M \subset \mathbb{R}^E$.

In the beginning of the proof for the circle C_n we showed that we can consider signs U_i instead of T. We prove a general version of that observation. Let us consider a random vector of signs $U \sim \mathcal{U}\left[\{\pm1\}^E\right]$ and a random subset $J \sim \mathcal{U}\left[2^E\right]$.

Lemma. $\mathrm{P}\left(\omega^T \in M\right) = \mathrm{P}\left(\omega_J \in M \mid \mathrm{NNC}\left(\omega_J\right)\right) = \mathrm{P}\left(U\omega \in M \mid \mathrm{NNC}\left(U\omega\right)\right)$

Proof. We compute a probability $\mathrm{P}\left(\mathrm{NNC}\left(U\omega\right)\right)$ at first. The probability is equal to a probability $\mathrm{P}\left(\mathrm{NNC}\left(x\right)\right)$ where $x \sim \mathcal{N}_G$ by theorem 6. And this probability is equal to $\mathrm{P}_{\mathrm{NNC}}\left(G\right)$, which is equal to $\frac{1}{|E|-|V|+1}$ by observation 16.

There is only a formal difference between $U\omega$ and ω_J. The weight function ω_J has opposite signs on edges which belongs into J, and $U\omega$ has opposite sign on edges e whenever $U_e = -1$. Random variables U a J have the same distribution functions. Hence, equalities $\mathrm{P}(\mathrm{NNC}(U\omega)) = \mathrm{P}(\mathrm{NNC}(\omega_J))$ and $\mathrm{P}(\omega_J \in M \mid \mathrm{NNC}(\omega_J)) = \mathrm{P}(U\omega \in M \mid \mathrm{NNC}(U\omega))$ hold.

Now, we have to prove the following equality.

$$\begin{aligned}
\mathrm{P}\left(\omega^T \in M\right) &= \mathrm{P}(\omega_J \in M \mid \mathrm{NNC}(\omega_J)) \\
\frac{1}{2^{|V|-1}} \int_{\omega \geq 0} \sum_{\substack{T \in \binom{V}{even} \\ \omega^T \in M}} \Phi^+(\omega)\,\mathrm{d}\omega &= \frac{1}{2^{|E|}\,\mathrm{P}(\mathrm{NNC}(\omega_J))} \int_{\omega \geq 0} \sum_{\substack{\mathrm{NNC}(\omega_J) \\ \omega_J \in M}} \Phi^+(\omega)\,\mathrm{d}\omega \\
\int_{\omega \geq 0} \sum_{\substack{T \in \binom{V}{even} \\ \omega^T \in M}} \Phi^+(\omega)\,\mathrm{d}\omega &= \int_{\omega \geq 0} \sum_{\substack{\mathrm{NNC}(\omega_J) \\ \omega_J \in M}} \Phi^+(\omega)\,\mathrm{d}\omega
\end{aligned}$$

By theorem 2 the weight function ω_J has no negative circle if, and only if, there exists $T \in \binom{V}{even}$ such that J is a minimal T-join. Minimal T-join is unique with probability one by proposition 8. Hence, a set of all ω for which internal sums are different has zero measure. □

Now, we have to prove that an equality $\mathrm{P}(U\omega \in M \mid \mathrm{NNC}(U\omega)) = \mathrm{P}(Z \in M)$ holds. Let us work with the left side of the equality

$$\mathrm{P}(U\omega \in M \mid \mathrm{NNC}(U\omega)) = \frac{\mathrm{P}(U\omega \in M, \mathrm{NNC}(U\omega))}{\mathrm{P}(\mathrm{NNC}(U\omega))} = \frac{\mathrm{P}(U\omega \in M')}{\mathrm{P}(\mathrm{NNC}(U\omega))}$$

We denote a set $\{\omega \in M \mid \mathrm{NNC}(M)\}$ by M'. As we discussed in the proof of the last lemma, the probability $\mathrm{P}(\mathrm{NNC}(U\omega))$ is equal to $2^{|V|-|E|-1}$.

$$\begin{aligned}
\mathrm{P}(U\omega \in M \mid \mathrm{NNC}(U\omega)) &= 2^{|E|+1-|V|}\,\mathrm{P}(U\omega \in M') \\
&= 2^{|E|+1-|V|} \sum_{U \in \{\pm 1\}^E} \int_{\substack{\omega \in \mathbb{R}^E \\ U\omega \in M'}} \frac{1}{2^{|E|}} \Phi^+(\omega)\,\mathrm{d}\omega \\
\text{using equation 1.1} & \\
&= 2^{|E|+1-|V|} \sum_{U \in \{\pm 1\}^E} \int_{\substack{\omega \geq 0 \\ U\omega \in M'}} \Phi(\omega)\,\mathrm{d}\omega
\end{aligned}$$

Let us express the right size of the equality $\mathrm{P}(U\omega \in M \mid \mathrm{NNC}(U\omega)) = \mathrm{P}(Z \in M)$.

$$\begin{aligned}
\mathrm{P}(Z \in M) &= \int_{z \in M} \Phi^G(z)\,\mathrm{d}z \\
&= \int_{z \in M'} \frac{\Phi(z)}{\mathrm{P}_{\mathrm{NNC}}(G)}\,\mathrm{d}z \\
&= 2^{|E|+1-|V|} \int_{z \in M'} \Phi(z)\,\mathrm{d}z
\end{aligned}$$

Now, it is sufficient to prove

$$\sum_{U\in\{\pm1\}^E} \int_{\substack{\omega\geq 0\\ U\omega\in M'}} \Phi(\omega)\,\mathrm{d}\omega = \int_{z\in M'} \Phi(z)\,\mathrm{d}z$$

which follows from lemma 3. ☐

3.3 Random join in a circle

In this section we show one example where our simpler way to generate the random join helps to compute the random join. The example is a circle $v_1, \dots, v_a, v_{a+1}, \dots, v_{a+b}$ which we shortly denote by C_{a+b}. We compute the probabilities of random joins between vertices v_a and v_{a+b}.

There are only two $v_a - v_{a+b}$ paths in the circle C_{a+b}. We denote the probability that the random join use a path $v_{a+b}, v_1, \dots, v_{a-1}, v_a$ and $v_{a+b}, v_{a+b-1}, \dots, v_{a+1}, v_a$ by P_a and P_b, respectively.

Let us compute the probability P_a. By theorem 8 we must compute

$$P_a = \mathrm{P}\left(\sum_{i=1}^{a} A_i < \sum_{i=1}^{b} B_i \;\middle|\; \sum_{i=1}^{a} A_i + \sum_{i=1}^{b} B_i > 0\right)$$

where $A_1, \dots, A_a$ and $B_1, \dots, B_b$ are independently chosen from the standard Gaussian distribution. By lemma 12 we know that $\sum_{i=1}^{a} A_i \sim \mathcal{N}(0, a)$ and $\sum_{i=1}^{b} B_i \sim \mathcal{N}(0, b)$. Hence, it is sufficient to compute

$$P_a = \mathrm{P}\left(x < y \mid x + y > 0\right)$$

where $x \sim \mathcal{N}(0, a)$ and $y \sim \mathcal{N}(0, b)$.

Now, it is time for integration.

$$\begin{aligned} P_a &= \mathrm{P}\left(x < y \mid x + y > 0\right) \\ &= 2\int_{\substack{x<y\\ x+y>0}} \phi_{0,a}(x)\phi_{0,b}(y)\,\mathrm{d}xy \\ &= 2\int_{\substack{x<y\\ x+y>0}} \frac{1}{2\pi\sqrt{ab}}\, e^{-\frac{1}{2}\left(\frac{x^2}{a}+\frac{y^2}{b}\right)}\,\mathrm{d}xy \end{aligned}$$

We use the ellipse substitution $\psi(r, \alpha) = (r\cos(\alpha)\sqrt{a},\, r\sin(\alpha)\sqrt{b})$. The integration after substitution is an integral over a set

$$\psi^{-1}(\{(x, y) \in \mathbb{R}^2 \mid x < y,\, x + y > 0\}) = \{(r, \alpha) \in \mathbb{R}^+ \times M\}$$

where $M = \left\{\alpha \in (-\pi, \pi) \;\middle|\; \cos(\alpha)\sqrt{a} < \sin(\alpha)\sqrt{b},\, \cos(\alpha)\sqrt{a} + \sin(\alpha)\sqrt{b} > 0\right\}$.

We can use the Fubini theorem after substitution.

$$\begin{aligned}
P_a &= \int_{\mathbb{R}^+ \times M} \frac{1}{\pi} r\, e^{-\frac{1}{2}r^2}\, \mathrm{d}r\alpha \\
&= \frac{1}{\pi} \int_M \int_{\mathbb{R}^+} r\, e^{-\frac{1}{2}r^2}\, \mathrm{d}r\, \mathrm{d}\alpha \\
&= \frac{1}{\pi} \int_M 1\, \mathrm{d}\alpha \\
&= \frac{1}{\pi} \lambda(M)
\end{aligned}$$

Now, we compute the measure of the set M. We need to express its two inequalities. Let us start by the first one.

$$\begin{aligned}
\cos(\alpha)\sqrt{a} &< \sin(\alpha)\sqrt{b} \\
\sqrt{\frac{a}{a+b}} \cos(\alpha) &< \sqrt{\frac{b}{a+b}} \sin(\alpha) \\
\sin(\beta)\cos(\alpha) &< \cos(\beta)\sin(\alpha) \\
\sin(\alpha - \beta) &> 0 \\
\alpha &\in (\beta, \pi + \beta)
\end{aligned}$$

We denoted $\sqrt{\frac{a}{a+b}}$ by $\sin(\beta)$. Readers can deduce that the equality $\cos(\beta) = \sqrt{\frac{b}{a+b}}$ holds. Then we used well-know equality

$$\sin(p \pm q) = \sin(p)\cos(q) \pm \cos(p)\sin(q).$$

We express the second inequality in the definition of the set M. The computing is similar so we shorten it.

$$\begin{aligned}
\cos(\alpha)\sqrt{a} + \sin(\alpha)\sqrt{b} &> 0 \\
\sin(\alpha + \beta) &> 0 \\
\alpha &\in (-\beta, \pi - \beta)
\end{aligned}$$

If we get those conditions together, we get that $M = (\beta, \pi - \beta)$. Hence,

$$P_a = \frac{1}{\pi}\lambda(M) = \frac{\pi - 2\beta}{\pi} = 1 - \frac{2}{\pi} \arcsin \sqrt{\frac{a}{a+b}}$$

and

$$P_b = \frac{2}{\pi} \arcsin \sqrt{\frac{a}{a+b}}.$$

From symmetry

$$P_a = \frac{2}{\pi} \arcsin \sqrt{\frac{b}{a+b}}.$$

We can accept the expressions as a result, but an arcsine of a square root of a fraction is too ugly for us. We do not know how we can get rid of the arcsine, but we can remove the square root from our expressions. We use an equality

$$\arcsin(p) \pm \arcsin(q) = \arcsin\left(p\sqrt{1-q^2} \pm q\sqrt{1-p^2}\right),$$

which we found in Bartsch [17].

Using this equation we get that $P_a + P_b = 1$, which is not a discovery for us but computing the difference is interesting.

$$P_a - P_b = \frac{2}{\pi} \arcsin \frac{b-a}{a+b}$$

When we sum up the last two equalities, we reach better formulas.

$$\begin{aligned} P_a &= \frac{1}{2\pi}\left(\pi + 2\arcsin\frac{b-a}{a+b}\right) \\ P_b &= \frac{1}{2\pi}\left(\pi - 2\arcsin\frac{b-a}{a+b}\right) \end{aligned}$$

Those formulas depend only on quotient of a and b. Let us denote $p = \frac{a}{a+b}$. If we substitute p into last two formulas we get

$$\begin{aligned} P_a &= \frac{1}{2\pi}\left(\pi + 2\arcsin(1-2p)\right) \\ P_b &= \frac{1}{2\pi}\left(\pi - 2\arcsin(1-2p)\right). \end{aligned}$$

Chapter 4

Algorithm

In this chapter we propose a special algorithm which finds the random join for a given weighted lattice and an even subset T of vertices.

We consider a connected weighted graph $G = (V, E, \omega)$ where ω is a non-negative weight function and an even set of vertices T in this chapter. The algorithm is optimised for the weighted lattice where a weight function is chosen from $\mathcal{N}_G^+$ but it works for a general connected graph.

4.1 Known algorithm

In this section we explain the known reduction a minimum T-join problem into a minimum weight perfect matching problem, and we show a list of known algorithms for the minimum weight perfect matching problem.

Let us describe the *minimum weight perfect matching problem*, which is significant for our T-join problem. A *matching* in a graph $G = (V, E)$ is a set M of such edges that no vertex of G is incident with more than one edge in M. Given a matching M, we say that M *covers* a vertex v (or that v is M-*covered*) if some edge of M is incident with V. Otherwise, v is M-*exposed*. A *maximum* matching is one of maximum cardinality. A *perfect* matching is one that covers all the vertices. Finally, we want a perfect matching having minimum weight with respect to some given edge-weights. The first problem is to decide whether a graph has a perfect matching. But we will solve this problem for a complete graph on even number of vertices, which always has a perfect matching.

In 1965, Edmonds [9] invented the famous *blossom-shrinking algorithm*, which solves the weighted perfect matching problem in polynomial time. A straightforward implementation of the blossom-shrinking algorithm, as originally proposed by Edmonds himself, requires time $O(n^2m)$, where n and m are the numbers of vertices and edges of G, respectively. Since then, the worst-case complexity of the blossom-shrinking algorithm has been improved successively: both Lawler [10] and Gabow [11] achieved a running time of $O(n^3)$. Galil, Micali and Gabow [13] improved the running time to $O(nm \log n)$ and finally Gabow [12] achieved a running time $O(n(m+n \log n))$. Somewhat better asymptotic running times are known for integral edge weights.

The minimum weight perfect matching problem is very deep and we will not study it. Our problem is how to reduce given minimum T-join problem into the minimum weight perfect matching problem. The well-known reduction is written in Cook's book [4]:

Step 1: Find the shortest $u-v$-path P_{uv} with respect to a weight function ω for each pair u, v of vertices from T. Let $d(u,v)$ be the weight of P_{uv}.

Step 2: Form a complete graph $G' = (T, E')$ with uv heaving weight $d(u,v)$ for each $uv \in E'$. Find a minimum-weight perfect matching M in G'.

Step 3: The symmetric difference of the edge-sets of paths P_{uv}, for $uv \in M$, is the minimum T-join.

4.2 Basic idea

Let us suppose that we would like to find the random join in a finite lattice which has at least 1000 rows and 100 columns for some experiments. If we use the classical reduction from the minimum T-join problem into the minimum weight perfect matching problem in a complete graph, we reach a graph which has approximately 10^5 vertices and 10^{10} edges. But this way is too time- and memory-consuming.

Fortunately, the weighted lattice has special properties. For example, for the most real-life problems the weights of edges satisfy the triangle non-equality. But every face but the external one in our lattice has 33% chance that one edge is heavier than the sum of weights of other edges of the face. It is easy to see that the heavier edge must not be a member of a minimal T-join so it can be removed from the lattice. The main idea of the algorithm consists in simplification of the lattice.

The algorithm is able to find the random join only in a small lattice, i.e. approximately 20×20 vertices. It is still necessary to find a minimal T-join in a graph which is much smaller. We recommend to use a reduction into minimum weight perfect matching proposed by Berman and Kahng and Vidhani and Zelikovsky [3] instead of the classical reduction described in previous section.

There are three useful operations on a graph $G = (V, E)$ which we use in the algorithm.

Deleting an edge e: $G \setminus e = (V, E \setminus \{e\})$ We simply remove the edge e.

Deleting a vertex v: $G \setminus v = G[V \setminus \{v\}]$ We remove the vertex v and all edges incident with v.

Contracting an edge $e = \{u, v\}$: We identify the vertices u and v and remove all resulting loops and duplicate edges. We denote the new graph by $G.e$.

4.3 Simple operations

We will describe several operations which simplify a given graph. Each operation modifies the weighted graph $G = (V, E, \omega)$ and the even set of vertices T into a weighted graph $G' = (V', E', \omega')$ and an even set of vertices T'. We say that an operation *decreases weights of T-joins by x* if there exists such one-to-one correspondence between T-joins in G and T'-joins in G' that for every T-join J in G and for the corresponding T'-join J' in G' holds $\omega(J) - x = \omega'(J')$. If $x = 0$, we say that the operation *retains weights of T-joins.*

In the previous section we mentioned the operation of removing a heavy edge. This operation does not retain all T-joins, but only minimal T-joins, which is sufficient for us. Formally, we say that an operation *decreases weights of minimum T-joins by x* if the one-to-one correspondence is only between minimal T-joins in G and minimal T'-joins in G'. The following definition and theorem explain what removing heavy edge exactly means.

Definition 4. *Edge e is called heavy edge of a circle C if $\omega(e) > \omega(C \setminus \{e\})$. Edge e is called heavy if there exists such circle C that edge e is a heavy edge of the circle C.*

Theorem 9. *Let edge e be a heavy edge of circle C. Then the edge e cannot be used in a minimal T-join, i.e. removing heavy edge retains weights of minimal T-joins.*

Proof. For contradiction we suppose that a minimal T-join J contains a heavy edge e of circle C. Let us denote $J' = J \triangle C$. J' is a T-join of weight

$$\begin{aligned}\omega(J') = \omega(J \triangle C) = \omega(J \setminus C) + \omega(C \setminus J) &\leq \omega(J \setminus \{e\}) + \omega(C \setminus \{e\}) \\ = \omega(J) - \omega(e) + \omega(C \setminus \{e\}) &< \omega(J).\end{aligned}$$

Which is contradiction to minimality of J. □

We could define that an edge e is a heavy edge of a circle C if $\omega(e) \geq \omega(C \setminus \{e\})$. When the equality holds and a minimal T-join J use the edge e, then $J \triangle C$ is also a minimal T-join. When we delete the edge e by theorem 9, then we omit the minimal T-join J. It is not a problem when we are looking for an arbitrary minimal T-join, but we could not say that removing heavy edge retains weight of minimal T-joins.

In our lattice we have four vertices of degree two and some vertices of degree three and a lot of vertices of degree four. When we remove all heavy edges from the lattice the number of vertices of small degree increases. We can easily remove vertices of degree one and two by following observations.

Let us start by removing vertices of degree one.

Theorem 10. *Let v be a vertex of degree one and $e = \{v, u\}$ be the unique edge. Denote $G' = G \setminus v$ and*

$$T' = \begin{cases} T & \text{if } v \notin T \\ T \triangle \{v, u\} & \text{otherwise.} \end{cases}$$

If $v \in T$ and $v \notin T$, then this operation decreases weights of T-joins by $\omega(e)$ and zero, respectively.

Proof. If $v \notin T$, then the edge e cannot be used in any T-join so we can use trivial one-to-one correspondence between T-joins to prove this theorem. If $v \in T$, then the edge e must be used in every T-join and the correspondence removes only the edge e from every T-join. □

Now, we let us see how we remove vertices of degree two. The removing depends on whether the vertex belongs into T or not.

Theorem 11. *Let us consider a vertex v of degree two which does not belong into T and the incident edges $e_1 = \{v, u_1\}$ and $e_2 = \{v, u_2\}$. Let G' be a graph $(V \setminus \{v\}, E' \setminus \{e_1, e_2\} \cup \{e\})$ where a new edge $e = \{u_1, u_2\}$ has weight $\omega(e_1) + \omega(e_2)$. Then this operation retains weights of T-joins.*

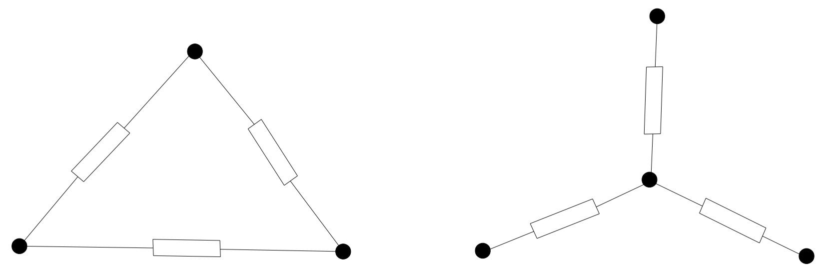

Figure 4.1: The triangle-star transfiguration in electronic

Proof. We know that both edges e_1 and e_2 or neither is used in every T-join. For given T'-join J' in G' we define a corresponding T-join J in G by formula

$$J = \begin{cases} J' & \text{if } e \notin J' \\ J' \cup \{e_1, e_2\} \setminus \{e\} & \text{otherwise.} \end{cases}$$

□

Actually, the last operation does not remove vertex v but it only contracts one of the edges incident to v. The following operation for a vertex which belongs to T is similar, but the weight of the new edge is different.

Theorem 12. *Let us consider vertex v of degree two which belongs to T and the incident edges $e_1 = \{v, u_1\}$ and $e_2 = \{v, u_2\}$ such that $\omega(e_1) \geq \omega(e_2)$. Let G' be a graph $(V \setminus \{v\}, E' \setminus \{e_1, e_2\} \cup \{e\})$ where a new edge $e = \{u_1, u_2\}$ has a weight $\omega(e_1) - \omega(e_2)$ and $T' = T \triangle \{v, v_1\}$. Then this operation decreases weights of T-joins by $\omega(e_2)$.*

Proof. We know that exactly one of the edges e_1 and e_2 is used in every T-join. For given T'-join J' in G' we define a corresponding T-join J in G by formula

$$J = \begin{cases} J' \cup \{e_2\} & \text{if } e \notin J' \\ J' \cup \{e_1\} \setminus \{e\} & \text{otherwise.} \end{cases}$$

□

It is possible that graph G already had edge $\{u_1, u_2\}$ and thus we get a multiple edge. One of the multiple edges must be heavy and we can remove it. But we have another way which is more useful in practice: A multiple edge can be created only by contracting an edge in a triangle, but we forbid any triangle. How we can forbid it? We change a triangle into a vertex of degree three by the following triangle-star transfiguration.

4.4 The triangle-star transfiguration

A theorem which is inspired with the triangle-star transfiguration from electronic (see figure 4.1) can be sometimes useful. We do not need any knowledge from physics. In this section we show how we can exchange a triangle and a star.

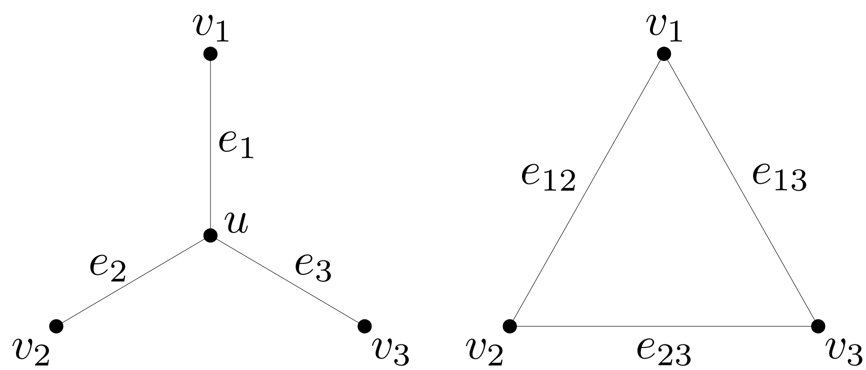

Figure 4.2: The triangle-star transfiguration

Theorem 13. *Let $u \notin T$ be a vertex of degree 3 and its adjacent edges $e_i = \{u, v_i\}$, for i = 1, 2, 3, having positive weights (see figure 4.2). Operation star-triangle transfiguration creates graph G' from graph G by removing vertex u and adding edges $e_{ij} = \{v_i, v_j\}$ of weight $\omega'(e_{ij}) = \omega(e_i) + \omega(e_j)$, for $1 \leq i < j \leq 3$. This operation retains weights of minimal T-joins.*

Proof. We know that exactly two of the star's edges, or none, belong into T-joins. If no edge of the star is in the T-join, then the corresponding T'-join does not contain any of triangle's edges. When exactly two edges e_i and e_j belong into the T-join, then the corresponding T'-join uses edge e_{ij}. The corresponding T-joins have the same weights.

We should also discuss that there is no other minimal T'-join in the graph G'. The graph G' have T'-joins J' which contain more than one triangle's edges. But $\omega'(J' \triangle C_3) < \omega'(J')$, where C_3 denotes the triangle because the triangle-inequality holds for the triangle. Hence, J' is not a minimal T'-join. $\square$

This transfiguration can be done in reverse.

Theorem 14. *We consider a triangle on vertices v_1 and v_2 and v_3. Let G' be a graph G after the triangle-star transfiguration which is reverse to the star-triangle transfiguration described in previous theorem. The weights of the star's edges are*

$$\omega'(e_i) = \frac{\omega(e_{ij}) + \omega(e_{ik}) - \omega(e_{jk})}{2},$$

for $\{i, j, k\} = \{1, 2, 3\}$. If the triangle inequality holds for weights of triangle's edges, then this operation also retains weights of minimal T-joins.

Proof. At the most one triangle's edge belongs into minimal T-join because the triangle inequality holds. If no edge belong into the T-join, then the corresponding T'-join has no star's edge. If edge e_{ij} is used in the T-join, then edges e_i and e_j belong into the corresponding T'-join. From equality

$$\omega'(e_i) + \omega'(e_j) = \frac{\omega(e_{ij}) + \omega(e_{ik}) - \omega(e_{jk})}{2} + \frac{\omega(e_{ij}) + \omega(e_{jk}) - \omega(e_{ik})}{2} = \omega(e_{ij})$$

it follows that the corresponding minimal T-joins have the same weights. $\square$

There is a condition that "the triangle inequality holds for weights of triangle's edges" in the last theorem. But this condition does not bother us because if it does not hold, then one of the triangle's edges is heavy and we can remove it.

If we transfigure a triangle into a star and the new star into a triangle, then we reach the same graph as we had before.

It is better store a graph with stars instead of triangles because our operations mostly prefer vertices of a smaller degree. As you will see in the following section, it is sometimes better to imagine that we have a triangle instead of a star.

4.5 Complex operations

In this section we show the operations on vertices of a degree three.

We say that a *couple of edges e and f is heavy* if there exists a circle C containing both edges e and f, and $\omega(e)+\omega(f) > \omega(C \setminus \{e, f\})$. Our lattice is created by tetragons. Every tetragon has six couples of edges - three pairs of complementary couples. Therefore, there exist three heavy couples of edges. That is the reason to study them.

Let us start with a vertex u of degree three such that $u \notin T$. Let us suppose that the couple of edges $e_i = \{u, v_i\}$ and $e_j = \{u, v_j\}$ is heavy on circle C, see figure 4.2. When we transfigure this star into a triangle, then an edge e_{ij} is heavy on the circle $C \setminus \{e_i, e_j\} \cup \{e_{ij}\}$. Hence, we may transfigure this star into a triangle and remove the edge e_{ij}.

What we can say about a similar situation in which the vertex u belongs into T? We cannot triangulate this star. We only know that both edges e_i and e_j, which create the heavy couple, cannot be used in minimal T-joins. But this fact is important as explained by the following lemma.

Lemma 4. *Let a vertex $u \in T$ of degree 3 be incident to edges e_1 and e_2 and e_3 and edges e_1 and e_2 belong into a circle C. Let x be a real number, which satisfies $0 < x \leq \min\{\omega(e_1), \omega(e_2), \omega(e_3)\}$ and*

$$\omega(e_1) + \omega(e_2) > \omega(C \setminus \{e_1, e_2\}) + 2x.$$

Let us denote $\omega'(e_i) = \omega(e_i) - x$, for $i = 1, 2, 3$. Then this operation decreases minimal T-joins by x.

Proof. Because $u \in T$, exactly one or three edges, which are incident with the vertex u, belong into a T-join. The couple of edges e_1 and e_2 is heavy in both graphs G and G' so this couple cannot be used in a minimal T-join in those graphs. Hence, exactly one edge belongs into a minimal T-join in both graphs and the trivial correspondence between minimal T-joins decreases their weights by x. $\square$

Increasing version of last lemma also holds when we change condition "a couple of edges e_1 and e_2 is heavy after decreasing" to "a couple of edges is heavy before increasing". But we did not find any situation where it was useful.

The last lemma is very useful when it holds for $x = \min\{\omega(e_1), \omega(e_2), \omega(e_3)\}$. When the edge e_3 has the smallest weight, then we can decrease weights by $x = \omega(e_3)$ and the edge e_3 gets a zero weight and it can be contracted by the following lemma.

Lemma 5. *Let the edge $e = \{u, v\}$ have a zero weight. Let $G' = G.e$ and*

$$T' = \begin{cases} T \setminus \{u, v\} & \text{if } u \in T \Leftrightarrow v \in T \\ T \setminus \{u, v\} \cup \{z\} & \text{otherwise} \end{cases}$$

where $z \notin V(G)$ means a vertex created by contracting the edge e. This operation retains weights of T-joins.

Proof. For given T-join J in the graph G a corresponding T'-join J' in the graph G' does not contain edge e and every edge in J which is adjacent with vertices u or v is replaced by a corresponding edge in the graph G'. □

But happens if the edge e_1 has the smallest weight among edges incident with the vertex u? From the inequality

$$\omega(e_2) = \omega(e_1) + \omega(e_2) - x > \omega(C \setminus \{e_1, e2\}) + 2x - x = \omega(C \setminus e_2) - \omega(e_1) + x = \omega(C \setminus e_2).$$

It follows that the edge e_1 is a heavy on the circle C. Similarly, if edge e_2 is the smallest, then the edge e_1 is heavy. Hence, this case is not interesting because theorem 9 covers it.

Corollary 2. *Let a vertex $u \in T$ of degree 3 be incident to edges e_1 and e_2 and e_3 and edges e_1 and e_2 belong into a circle C. Suppose that $\omega(e_3) \leq \omega(e_1)$ and $\omega(e_3) \leq \omega(e_2)$ and*

$$2(\omega(e_1) + \omega(e_2) - \omega(e_3)) > \omega(C).$$

If we decrease weights of edges e_1 and e_2 by $\omega(e_1)$ and we contract the edge e_3, then we decrease minimal T-joins by $\omega(e_3)$.

4.6 Implementation notices

There is a natural question whether the main theorem 8 is useful in experiments, i.e. whether it is possible to generate the random join so that we choose the weight function ω from $\mathcal{N}_G^{NNC}$ and we find the shortest path with respect to ω.

There are two important problems: How do we generate the weight function and how do we find the shortest path.

Generating the weight function by the definition of $\mathcal{N}_G^{NNC}$ is too time-consuming. The probability that a weight function chosen from $\mathcal{N}_G$ has no negative circle is $2^{\frac{1}{|E|-|V|+1}}$ by proposition 16.

We cannot use Dijkstra's algorithm to find the shortest path because Dijkstra's algorithm requests non-negative weights. Fortunately, the weight function has no negative circle otherwise the shortest path problem is NP-complete. The Bellman-Ford algorithm is suitable for our problem. It is described in Wikipedia [18] and in a book [19]. The time complexity of the Bellman-Ford algorithm is $|V||E|$.

Another question is whether it is necessary to find two minimal T-join, as requested by the definition of the random join. It is possible to find both minimal T-joins at once by our algorithm! We can save both T_1 and T_2 as bit-mask in each vertex. We have at most two vertices which belong into exactly one of sets T_1 and T_2. We cannot do some operations on those vertices because the results of those operations depend on T. But it does not worry so much because our operations make only local changes.

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