Matchings extend into 2-factors in hypercubes

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Abstract

Vandenbussche and West conjectured that every matching of the hypercube can be extended to a 2-factor. We prove this conjecture.

1 Introduction

A set of edges $P \subset E$ of a graph $G = (V, E)$ is a *matching* if every vertex of G is incident with at most one edge of P. If a vertex v of G is incident with an edge of P, we say that v is *covered* by P. A matching P is *perfect* if every vertex of G is covered by P. A set of edges $S \subseteq E$ is called *k-factor* if every vertex of the subgraph (V, S) has degree exactly k. Clearly, 1-factors are exactly perfect matchings. Next, a 2-factor is a union of vertex-disjoint cycles covering all vertices. If a 2-factor forms a single cycle, then it is called a *Hamiltonian cycle*.

The *n-dimensional hypercube* Q_n is a graph whose vertex set consists of all binary vectors of length n, with two vertices being adjacent whenever the corresponding vectors differ at exactly one coordinate. It is well known that Q_n is Hamiltonian for every $n \geq 2$. This statement can be traced back to 1872 [6]. Since then the research on Hamiltonian cycles in hypercubes satisfying certain additional properties has received considerable attention. Dvořák [3] showed that any set of at most $2n - 3$ edges of Q_n $(n \geq 2)$ that induces vertex-disjoint paths is contained in a Hamiltonian cycle.

Ruskey and Savage [10] asked whether every matching of the hypercube Q_n can be extended into a Hamiltonian cycle and this problem is still open. One natural step toward this problem is considering perfect matchings only. Kreweras [9] conjectured that every perfect matching in the n-dimensional hypercube with $n \geq 2$ extends to a Hamiltonian cycle. This conjecture was popularized by Knuth [8] and proven by Fink [4]. The proof of Kreweras' conjecture actually provides a slightly stronger statement saying that every matching in the complete graph on vertices of Q_n can be extended into a Hamiltonian cycle using only edges of Q_n [4]. This result inspired several generalizations [1, 5], e.g. the authors of [1] showed that Kreweras' conjecture also holds for sparse spanning regular subgraphs of hypercubes. Dimitrov et al. [2] presented a complementary result that the hypercube Q_n contains a Hamiltonian cycle avoiding a given matching except a forbidden configuration. An interested reader can find more details about this topic in the survey of Savage [11].

Another, weaker form of the problem of Ruskey and Savage was considered by Vandenbussche and West [12] who conjectured that every matching can be extended into a 2-factor.

Conjecture 1.1 (Vandenbussche, West [12])**.** *Every matching of the hypercube Q_n can be extended into a 2-factor where $n \geq 2$.*

Vandenbussche and West [12] verified the conjecture for dimension n at most 5. In this paper, we prove that the conjecture holds.

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2 The proof

The weight $|u|$ of a vertex u of Q_n is the number of 1's in the binary vector u and $\text{parity}(u) = |u| \bmod 2$. A vertex u of Q_n is called even if $\text{parity}(u) = 0$ and odd otherwise. We consider the canonical orientation of all edges of Q_n such that every edge is oriented from its even endvertex to the odd one.

A subgraph Q of the hypercube Q_n is called a *subcube of dimension* d if Q is isomorphic to the d-dimensional hypercube Q_d where $1 \le d \le n$. In this paper, we consider subcubes of dimension 2 only. A coordinate of an edge uv of Q_n is the coordinate in which the binary vectors u and v differ, denoted by $u \triangle v$. The graph obtained from Q_n by removing all edges in coordinates $3, \dots, n$ consists of 2^{n-2} components forming 2-dimensional subcubes with edges in the first and the second coordinate, and let $\mathcal{C}$ be the set of all these subcubes. Given $\mathcal{C}$ and a set of edges $S \subseteq Q_n$ the interconnection graph $I(\mathcal{C}, S)$ is the oriented multigraph where every subcube of $\mathcal{C}$ is represented by a single vertex and two vertices of $I(\mathcal{C}, S)$ are connected by as many edges as there are edges of S between corresponding subcubes while preserving orientations of edges; see Figure 1. A *reverse* of an oriented multigraph is obtained by reversing the orientation of all edges. The degree $\deg_S(Q)$ of a subcube $Q \in \mathcal{C}$ is the number of edges of S having exactly one endvertex in Q. Furthermore, $\text{indeg}_S(Q)$ is the number of edges of S incoming to Q from other subcubes, and similarly, $\text{outdeg}_S(Q)$ is the number of edges of S outgoing from Q to other subcubes. Note that $\deg_S(Q)$, $\text{indeg}_S(Q)$ and $\text{outdeg}_S(Q)$ are the appropriate degrees of the vertex corresponding to Q in $I(\mathcal{C}, S)$. Whenever we discuss components, paths, or cycles in an oriented multigraph, we neglect the orientation of edges, so e.g. orientations of edges on a cycle may alternate.

Our goal is to find a set of edges R of Q_n which extends a given matching P into a 2-factor. In order to avoid confusion, we require P and R to be disjoint. Hence, $P \cup R$ is a 2-factor if and only if $P \cap R = \emptyset$ and every vertex of Q_n covered by P is incident with exactly one edge of R and every vertex of Q_n uncovered by P is incident with exactly two edges of R.

We prove Conjecture 1.1 using the following lemma.

Lemma 2.1. *Let P be a matching of Q_n with $n \ge 2$ such that all edges between every two vertices of $I(\mathcal{C}, P)$ have the same orientation. Then, there exists a set of edges R of Q_n such that $P \cup R$ is a 2-factor of Q_n and $P \cap R = \emptyset$ and*

$$I(\mathcal{C}, R) \text{ equals the reverse of } I(\mathcal{C}, P). \tag{1}$$

First, we show how Conjecture 1.1 follows from this lemma.

Theorem 2.2. *For every matching P of Q_n with $n \ge 2$ there exists a set of edges R of Q_n such that the union $P \cup R$ forms a 2-factor.*

Proof. We convert the matching P of Q_n into a matching P' of Q_n satisfying the assumptions of Lemma 2.1 which provides us a set of edges R' of Q_n extending P' into a 2-factor of Q_n. Then, we convert R' into a set of edges of R of Q_n such that $P \cup R$ is a 2-factor as this theorem requires. We present simple rules how to construct P' from P and, after the application of Lemma 2.1, how to construct R from R'. These rules are applied to every pair of subcubes of $\mathcal{C}$ and they modify sets P' and R. In the beginning, we initialize $P' := P$ and after the application of Lemma 2.1 we initialize $R := R'$.

Now, we present the rules to modify P' and R. We process every pair of subcubes Q and Q' of $\mathcal{C}$ having edges of P between Q and Q' in both directions as follows. Since we consider subcubes of dimension 2, the hypercube Q_n contains exactly two edges between Q and Q' in each direction. We distinguish the following cases.

1. If P contains 3 edges between Q and Q', then P contains exactly one edge uv in one direction, say from Q to Q'. In this case, we remove the edge uv from P'. The extending set of edges R' has to contain both edges from Q' to Q to ensure (1). In the construction of R from R', we remove the edge uv from R as it is already contained in P which guarantees that P and R are disjoint.

2. If P contains exactly one edge uu' from Q to Q' and exactly one edge $v'v$ from Q' to Q, then we replace the edges uu' and $v'v$ by the edges uv and $v'u'$ in P', so P' has no edge between Q and Q'. From (1) it follows that there is also no edge between Q and Q' in R' and we also let R have no edges between Q and Q'.

3. If P contains all 4 edges between Q and Q', then we replace these 4 edges by two non-adjacent edges of Q and two non-adjacent edges of Q' in P', so P' has no edge between Q and Q'. Then R' has no edges between Q and Q', and we also let R have no edge between Q and Q'.

This way we process every pair of subcubes once to construct P' and then once more to reconstruct R. Furthermore, the construction of P' ensures that P' does not contain edges of both directions between any two subcubes, so Lemma 2.1 can be applied. Then, the construction of R ensures that degrees of all vertices of Q_n in $P \cup R$ are the same as in $P' \cup R'$ and also $P \cap R = P' \cap R' = \emptyset$. □

Now, we prove the lemma.

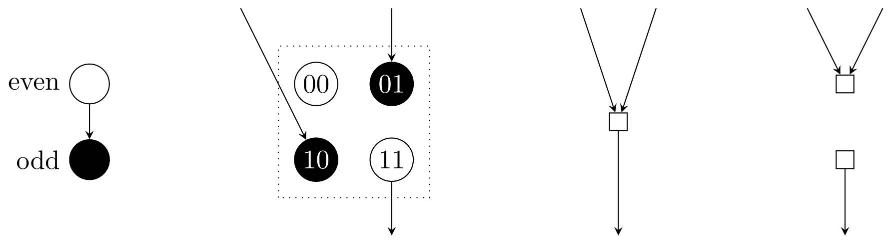

Figure 1: Canonical orientation of edges; a subcube with 2 incoming and 1 outgoing edges of P; the corresponding vertex in $I(\mathcal{C}, P)$; the crossroad split into two vertices in $L(\mathcal{C}, P)$. The two-digit numbers denote the values in the first two coordinates of vertices.

Proof of Lemma 2.1. A subcube $Q \in \mathcal{C}$ with $\operatorname{indeg}_P(Q) = 2$ or $\operatorname{outdeg}_P(Q) = 2$ is called a *crossroad*. Let $L(\mathcal{C}, P)$ be a graph obtained from $I(\mathcal{C}, P)$ by splitting every crossroad Q into two vertices where one is incident with all edges of P incoming to Q and the other is incident with all edges of P outgoing from Q; see Figure 1. Note that non-crossroads of $I(\mathcal{C}, P)$ are unchanged in $L(\mathcal{C}, P)$. Observe that every vertex of $L(\mathcal{C}, P)$ has degree at most two, so every component of $L(\mathcal{C}, P)$ is a path or a cycle or an isolated vertex or two parallel edges between two crossroads.

Consider a cycle D of $L(\mathcal{C}, P)$. The orientation of edges along the cycle D alternates in every crossroad and is preserved in every non-crossroad. Since the orientation of edges in the cycle D alternates even times, the cycle D contains an even number of crossroads, as well. Furthermore, the cycle D corresponds to a closed walk of Q_{n-2} so the cycle D has even length. Therefore, every cycle of $L(\mathcal{C}, P)$ contains an even number of non-crossroad vertices.

Now, we colour every edge e of $L(\mathcal{C}, P)$ by a colour $c(e) \in \{1, 2\}$ so that for every pair of adjacent edges e and e' sharing a common endvertex Q satisfies

$$c(e) = c(e') \text{ if and only if } Q \text{ is a crossroad.} \tag{2}$$

Note that parallel edges e and e' sharing both endvertices have the same colour by (2) since their endvertices are crossroads. Furthermore, note that every non-crossroad Q has at most two incident edges in $L(\mathcal{C}, P)$, and they have opposite colours by (2). Every component of $L(\mathcal{C}, P)$ forming a path can be greedily coloured to satisfy (2). Similarly, every component of $L(\mathcal{C}, P)$ forming a cycle can be greedily coloured to satisfy (2) since every cycle of $L(\mathcal{C}, P)$ contains an even number of non-crossroads, so the colour is alternated even times along the cycle.

Next, we describe all edges of R between different subcubes of $\mathcal{C}$. Consider an edge xx' of P from a subcube Q to another subcube Q'. According to (1), R has to contain an edge $y'y$ from

Q' to Q in the direction opposite to xx'. Here $y'y$ can be chosen from two such edges between Q and Q' since $x \triangle y$ is either the first or the second coordinate. We choose

$$\text{the coordinate } x \triangle y \text{ to be the colour } c(xx'). \tag{3}$$

Note that Q' contains a unique vertex y' such that $y'y$ is an edge of Q_n and the coordinate $x' \triangle y'$ is also $c(xx')$. We add this edge to R. Furthermore, if P contains two edges uu' and vv' between a pair of subcubes Q and Q', then these edges have the same orientation by the assumption of the lemma. Hence, both Q and Q' are crossroads so $c(uu') = c(vv')$ and thus R contains the remaining two edges between Q of Q' of opposite direction. This R clearly satisfies (1).

Finally, we describe all edges of R inside the subcubes of $\mathcal{C}$ assuming that R already contains the edges between different subcubes as presented above. Consider a subcube $Q \in \mathcal{C}$ and let a, b, c and d be all vertices of Q so that a and c are the odd vertices. Without loss of generality, we assume that $\text{indeg}_P(Q) \geq \text{outdeg}_P(Q)$ and we distinguish the following cases. It is easy to check in all the following cases that every vertex of Q will have two incident edges in $P \cup R$ and no edge of Q will be contained in both P and R, which implies that this lemma holds.

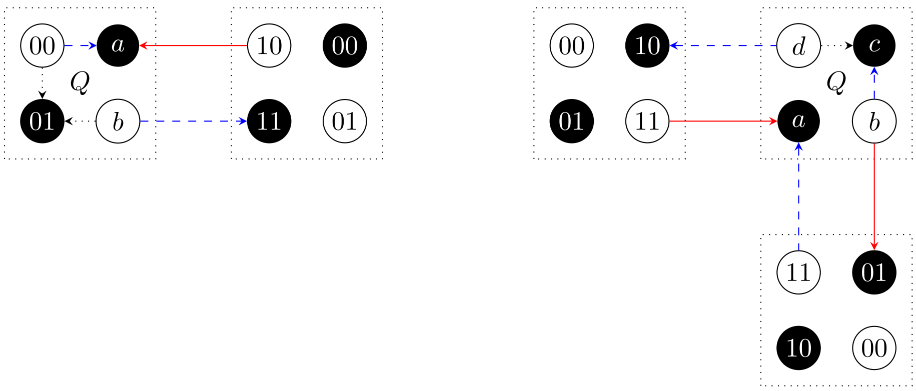

Figure 2: 2-factor in a non-crossroad Q with $\text{indeg}_P(Q) = 1$ and $\text{outdeg}_P(Q) \in \{0, 1\}$, i.e. cases 1b (the left figure) and 1c (the right figure) of the proof of Lemma 2.1. Full red lines are edges of P and dashed blue lines are edges of R and dotted black lines belong either to P or R.

1. Assume that Q is a non-crossroad; see Figure 2 for cases (b) and (c).

   (a) $\text{indeg}_P(Q) = \text{outdeg}_P(Q) = 0$. In this case, we add all edges of Q not contained in P into R.

   (b) $\text{indeg}_P(Q) = 1$ and $\text{outdeg}_P(Q) = 0$. P covers one odd vertex (say a) of Q by an incoming edge and by (1) R covers one even vertex (say b) of Q by an outgoing edge. So, we add edges $E(Q) \setminus (P \cup \{ba\})$ into R.

   (c) $\text{indeg}_P(Q) = \text{outdeg}_P(Q) = 1$. Assume that a and b are the vertices of Q covered by edges $a'a$ and bb' of P incoming to Q and outgoing from Q, respectively. By (1), Q contains vertices x and y already covered by edges of R incoming to Q and outgoing from Q, respectively. From (2) it follows that edges of $L(\mathcal{C}, P)$ corresponding to $a'a$ and bb' have the opposite colour, so (3) implies $a \triangle y \neq b \triangle x$. Furthermore, a and y are neighbour vertices as well as b and x which implies $|\{a, y\} \cap \{b, x\}| = 1$. From parities of all vertices it follows that either $x = a$ or $y = b$ and without loss of generality we assume that $x = a$ which implies $y = d$. We add the edge bc into R and we also add the edge cd into R unless cd is already contained in P.

2. Assume that Q is a crossroad; see Figure 3. Since $\text{indeg}_P(Q) \geq \text{outdeg}_P(Q)$ it follows that $\text{indeg}_P(Q) = 2$. Hence, P covers both odd vertices of Q by incoming edges and P contains no edge of Q which simplifies the proof since it is impossible to fail the condition that no

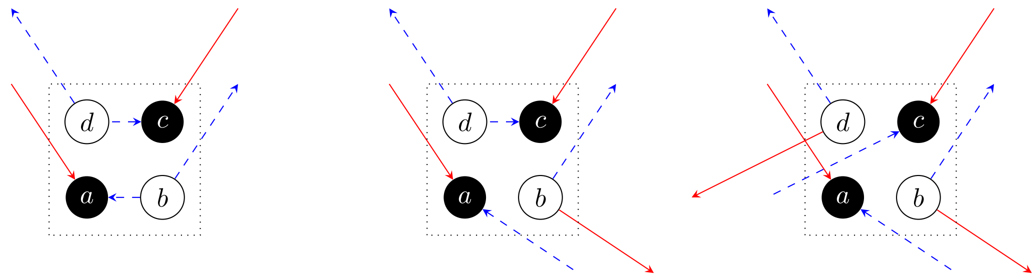

Figure 3: 2-factor in a crossroad Q with $\mathrm{indeg}_P(Q) = 2$ and $\mathrm{outdeg}_P(Q) \in \{0, 1, 2\}$.

edge of Q is contained in both P and R. Furthermore, by (1) R contains two edges outgoing from Q.

(a) $\mathrm{outdeg}_P(Q) = 0$. We add edges ab and cd into R.

(b) $\mathrm{outdeg}_P(Q) = 1$. P covers one even vertex (say b) of Q by an outgoing edge and by (1) R covers one odd vertex (say a) by an incoming edge. We add the edge cd into R.

(c) $\mathrm{outdeg}_P(Q) = 2$. P covers both even vertices of Q by incoming edges and R covers both odd vertices of Q by outgoing edges. This implies that every vertex of Q has two incident edges in $P \cup R$, so no edge needs to be added into R.

□

3 Concluding remarks

Note that the following conclusions trivially follow from Theorem 2.2 using the well known fact that the edges of every regular bipartite graph may be partitioned into perfect matchings [7].

Corollary 3.1. *Every matching of the hypercube Q_n can be extended into a k-factor where $n \geq k \geq 2$.*

Corollary 3.2. *For every matching P of Q_n there exists a k-factor R of Q_n avoiding P where $n - 2 \geq k \geq 1$.*

In this paper, we proved that every matching of Q_n can be extended into a 2-factor. However, the presented construction finds a 2-factor which may contain up to 2^{n-2} cycles, e.g. when P contains all edges of Q_n of the first coordinate. We are interested in an improved construction which significantly reduces the number of cycles in a 2-factor, ideally to a single one [10].

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