

Optimization methods

NOPT048

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Plan of the lecture

- Linear and integer optimization
- Convex sets and Minkowski-Weyl theorem
- Simplex methods
- Duality of linear programming
- Ellipsoid method
- Unimodularity
- Minimal weight maximal matching
- Matroid
- Cut and bound method

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Literature

- A. Schrijver, Theory of linear and integer programming, John Wiley, 1986
- W. J. Cook, W. H. Cunningham, W. R. Pulleyblank, A. Schrijver, Combinatorial Optimization, John Wiley, 1997
- J. Matoušek, B. Gärtner, Understanding and using linear programming, Springer, 2006.
- J. Matoušek Introduction to Discrete Geometry. ITI Series 2003-150, MFF UK, 2003

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Optimization

Mathematical optimization

is the selection of a best element (with regard to some criteria) from some set of available alternatives.

Examples

- Minimize $x^2 + y^2$ where $(x, y) \in \mathbb{R}^2$
- Maximal matching in a graph
- Minimal spanning tree
- Shortest path between given two vertices

Optimization problem

Given a set of solutions M and an objective function $f : M \rightarrow \mathbb{R}$, optimization problem is finding a solution $x \in M$ with the maximal (or minimal) objective value $f(x)$ among all solutions of M .

Duality between minimization and maximization

If $\min_{x \in M} f(x)$ exists, then also $\max_{x \in M} -f(x)$ exists and $-\min_{x \in M} f(x) = \max_{x \in M} -f(x)$.

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Content

- 1 Linear programming
- 2 Linear, affine and convex sets
- 3 Simplex method
- 4 Duality of linear programming
- 5 Integer linear programming
- 6 Matching
- 7 Ellipsoid method
- 8 Vertex Cover
- 9 Matroid

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General information

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Consultations Individual schedule

Examination

- Tutorial conditions
 - Tests
 - Theoretical homeworks
 - Practical homeworks
- Pass the exam

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Outline

- 1 Linear programming
- 2 Linear, affine and convex sets
- 3 Simplex method
- 4 Duality of linear programming
- 5 Integer linear programming
- 6 Matching
- 7 Ellipsoid method
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Notation: Vector and matrix

Matrix

A matrix of type $m \times n$ is a rectangular array of m rows and n columns of real numbers. Matrices are written as A, B, C , etc.

Vector

A vector is an n -tuple of real numbers. Vectors are written as $\mathbf{c}, \mathbf{x}, \mathbf{y}$, etc. Usually, vectors are column matrices of type $n \times 1$.

Scalar

A scalar is a real number. Scalars are written as a, b, c , etc.

Special vectors

$\mathbf{0}$ and $\mathbf{1}$ are vectors of zeros and ones, respectively.

Transpose

The transpose of a matrix A is matrix A^T created by reflecting A over its main diagonal. The transpose of a column vector \mathbf{x} is the row vector \mathbf{x}^T .

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Elements of a vector and a matrix

- The i -th element of a vector \mathbf{x} is denoted by x_i .
- The (i, j) -th element of a matrix A is denoted by $A_{i,j}$.
- The i -th row of a matrix A is denoted by $A_{i,*}$.
- The j -th column of a matrix A is denoted by $A_{*,j}$.

Dot product of vectors

The dot product (also called inner product or scalar product) of vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ is the scalar $\mathbf{x}^T \mathbf{y} = \sum_{i=1}^n x_i y_i$.

Product of a matrix and a vector

The product $A\mathbf{x}$ of a matrix $A \in \mathbb{R}^{m \times n}$ of type $m \times n$ and a vector $\mathbf{x} \in \mathbb{R}^n$ is a vector $\mathbf{y} \in \mathbb{R}^m$ such that $y_i = A_{i,*} \mathbf{x}$ for all $i = 1, \dots, m$.

Product of two matrices

The product AB of a matrix $A \in \mathbb{R}^{m \times n}$ and a matrix $B \in \mathbb{R}^{n \times k}$ a matrix $C \in \mathbb{R}^{m \times k}$ such that $C_{*,j} = AB_{*,j}$ for all $j = 1, \dots, k$.

Example of linear programming: Optimized diet

Express using linear programming the following problem

Find the cheapest vegetable salad from carrots, white cabbage and cucumbers containing required amount the vitamins A and C and dietary fiber.

Food	Carrot	White Cabbage	Cucumber	Required per meal
Vitamin A [mg/kg]	35	0.5	0.5	0.5 mg
Vitamin C [mg/kg]	60	300	10	15 mg
Dietary Fiber [g/kg]	30	20	10	4 g
Price [EUR/kg]	0.75	0.5	0.15	

Formulation using linear programming

Let $\mathbf{x}_1, \mathbf{x}_2$ and \mathbf{x}_3 be real variables denoting the amount of carrots, white cabbage and cucumbers, respectively. The linear programming problem is

$$\begin{array}{ll}
 \text{Minimize} & 0.75x_1 + 0.5x_2 + 0.15x_3 \\
 \text{subject to} & 35x_1 + 0.5x_2 + 0.5x_3 \geq 0.5 \\
 & 60x_1 + 300x_2 + 10x_3 \geq 15 \\
 & 30x_1 + 20x_2 + 10x_3 \geq 4 \\
 & x_1, x_2, x_3 \geq 0
 \end{array}$$

Example of integer linear programming: Vertex cover

Vertex cover problem

Given undirected graph (V, E) , find the smallest set of vertices $U \subseteq V$ covering every edge of E ; that is, $U \cup e \neq \emptyset$ for every $e \in E$.

Formulation using integer linear programming

Variables: cover $x_v \in \{0, 1\}$ for every vertex $v \in V$

Covering: $x_u + x_v \geq 1$ for every edge $uv \in E$

Objective function: Minimize $\mathbf{1}^T \mathbf{x}$

Related problems

Integer linear programming

Integer linear programming problem is an optimization problem to find $\mathbf{x} \in \mathbb{Z}^n$ which maximizes $\mathbf{c}^T \mathbf{x}$ and satisfies $A\mathbf{x} \leq \mathbf{b}$ where $A \in \mathbb{R}^{m \times n}$ and $\mathbf{b} \in \mathbb{R}^m$.

Mix integer linear programming

Some variables are integer and others are real.

Binary linear programming

Every variable is either 0 or 1. ①

Complexity

- A linear programming problem is efficiently solvable, both in theory and in practice.
- The classical algorithm for linear programming is the *Simplex method* which is fast in practice but it is not known whether it always run in polynomial time.
- Polynomial time algorithms the *ellipsoid* and the *interior point* methods.
- No strongly polynomial-time algorithms for linear programming is known.
- Integer linear programming is NP-hard.

Equality and inequality of two vectors

For vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ we denote

- $\mathbf{x} = \mathbf{y}$ if $x_i = y_i$ for every $i = 1, \dots, n$ and
- $\mathbf{x} \leq \mathbf{y}$ if $x_i \leq y_i$ for every $i = 1, \dots, n$.

System of linear equations

Given a matrix $A \in \mathbb{R}^{m \times n}$ of type $m \times n$ and a vector $\mathbf{b} \in \mathbb{R}^m$, the formula $A\mathbf{x} = \mathbf{b}$ means a system of m linear equations where \mathbf{x} is a vector of n real variables.

System of linear inequalities

Given a matrix $A \in \mathbb{R}^{m \times n}$ of type $m \times n$ and a vector $\mathbf{b} \in \mathbb{R}^m$, the formula $A\mathbf{x} \leq \mathbf{b}$ means a system of m linear inequalities where \mathbf{x} is a vector of n real variables.

Example: System of linear inequalities in two different notations

$$\begin{array}{rclcl}
 2x_1 & + & x_2 & + & x_3 & \leq & 14 \\
 2x_1 & + & 5x_2 & + & 5x_3 & \leq & 30
 \end{array}
 \quad
 \begin{pmatrix} 2 & 1 & 1 \\ 2 & 5 & 5 \end{pmatrix}
 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}
 \leq
 \begin{pmatrix} 14 \\ 30 \end{pmatrix}$$

Example of linear programming: Network flow

Network flow problem

Given directed graph (V, E) with capacities $\mathbf{c} \in \mathbb{R}^E$ and a source $s \in V$ and a sink $t \in V$, find the maximal flow from s to t satisfying the flow conservation and capacity constraints.

Formulation using linear programming

Variables: flow f_e for every edge $e \in E$

Capacity constraints: $0 \leq f \leq c$

Flow conservation: $\sum_{uv \in E} f_{uv} = \sum_{vw \in E} f_{vw}$ for every $v \in V \setminus \{s, t\}$

Objective function: Maximize $\sum_{sw \in E} f_{sw} - \sum_{us \in E} f_{us}$

Linear Programming

Canonical form

Linear programming problem in the canonical form is an optimization problem to find $\mathbf{x} \in \mathbb{R}^n$ which maximizes $\mathbf{c}^T \mathbf{x}$ and satisfies $A\mathbf{x} \leq \mathbf{b}$ where $A \in \mathbb{R}^{m \times n}$ and $\mathbf{b} \in \mathbb{R}^m$.

Equation form

Linear programming problem in the equation form is a problem to find $\mathbf{x} \in \mathbb{R}^n$ which maximizes $\mathbf{c}^T \mathbf{x}$ and satisfies $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$ where $A \in \mathbb{R}^{m \times n}$ and $\mathbf{b} \in \mathbb{R}^m$.

Conversions

- Every $\mathbf{x} \in \mathbb{R}^n$ satisfies $A\mathbf{x} = \mathbf{b}$ if and only if it satisfies $A\mathbf{x} \geq \mathbf{b}$ and $A\mathbf{x} \leq \mathbf{b}$.
- Every $\mathbf{x} \in \mathbb{R}^n$ satisfies $A\mathbf{x} \leq \mathbf{b}$ if and only if there exists $\mathbf{z} \in \mathbb{R}^m$ satisfying $A\mathbf{x} + \mathbf{z} = \mathbf{b}$ and $\mathbf{z} \geq \mathbf{0}$.
- Every occurrence of a variable x can be replaced by $x^+ - x^-$ when contains $x^+, x^- \geq 0$ are added.

Example: Conversion from the canonical form into the equation form

- max $\mathbf{c}^T \mathbf{x}$ such that $A\mathbf{x} \leq \mathbf{b}$
- max $\mathbf{c}^T \mathbf{x}$ such that $A\mathbf{x} + \mathbf{z} = \mathbf{b}$ and $\mathbf{z} \geq \mathbf{0}$
- max $\mathbf{c}^T \mathbf{x}^+ - \mathbf{c}^T \mathbf{x}^-$ such that $A\mathbf{x}^+ - A\mathbf{x}^- + \mathbf{z} = \mathbf{b}$ and $\mathbf{z}, \mathbf{x}^+, \mathbf{x}^- \geq \mathbf{0}$

- Show that binary linear programming is a special case of integer linear programming.

Basic terminology

- Number of variables: n
- Number of constraints: m
- Solution: \mathbf{x}
- Objective function: e.g. $\max \mathbf{c}^T \mathbf{x}$
- Feasible solution: a solution satisfying all constraints, e.g. $A\mathbf{x} \geq \mathbf{b}$
- Optimal solution: a feasible solution maximizing $\mathbf{c}^T \mathbf{x}$
- Infeasible problem: a problem having no feasible solution
- Unbounded problem: a problem having a feasible solution with arbitrary large value of given objective function
- Polyhedron: a set of points $\mathbf{x} \in \mathbb{R}^n$ satisfying $A\mathbf{x} \geq \mathbf{b}$ for some $A \in \mathbb{R}^{m \times n}$ and $\mathbf{b} \in \mathbb{R}^m$
- Polytope: a bounded polyhedron

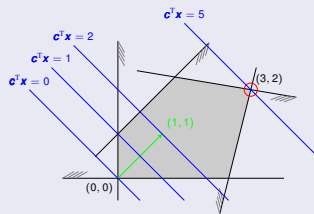
Graphical method: Optimal solution

Example

Find the optimal solution of the following problem.

$$\begin{array}{llll} \text{Maximize} & \mathbf{x}_1 & + & \mathbf{x}_2 \\ & \mathbf{x}_1 & + & 6\mathbf{x}_2 \leq 15 \\ & 4\mathbf{x}_1 & - & \mathbf{x}_2 \leq 10 \\ & -\mathbf{x}_1 & + & \mathbf{x}_2 \leq 1 \\ & \mathbf{x}_1, \mathbf{x}_2 & \geq & 0 \end{array}$$

Solution



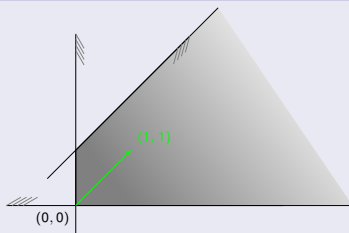
Graphical method: Unbounded problem

Example

Show that the following problem is unbounded.

$$\begin{array}{llll} \text{Maximize} & \mathbf{x}_1 & + & \mathbf{x}_2 \\ & -\mathbf{x}_1 & + & \mathbf{x}_2 \leq 1 \\ & \mathbf{x}_1, \mathbf{x}_2 & \geq & 0 \end{array}$$

Solution



Outline

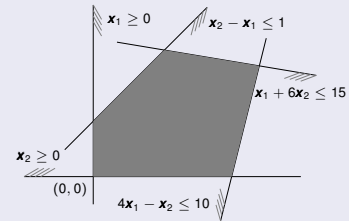
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Example

Draw the set of all feasible solutions $(\mathbf{x}_1, \mathbf{x}_2)$ satisfying the following conditions.

$$\begin{array}{llll} \mathbf{x}_1 & + & 6\mathbf{x}_2 & \leq 15 \\ 4\mathbf{x}_1 & - & \mathbf{x}_2 & \leq 10 \\ -\mathbf{x}_1 & + & \mathbf{x}_2 & \leq 1 \\ \mathbf{x}_1, \mathbf{x}_2 & \geq & 0 \end{array}$$

Solution



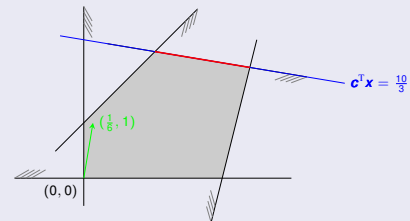
Graphical method: Multiple optimal solutions

Example

Find all optimal solutions of the following problem.

$$\begin{array}{llll} \text{Maximize} & \frac{1}{6}\mathbf{x}_1 & + & \mathbf{x}_2 \\ & \mathbf{x}_1 & + & 6\mathbf{x}_2 \leq 15 \\ & 4\mathbf{x}_1 & - & \mathbf{x}_2 \leq 10 \\ & -\mathbf{x}_1 & + & \mathbf{x}_2 \leq 1 \\ & \mathbf{x}_1, \mathbf{x}_2 & \geq & 0 \end{array}$$

Solution



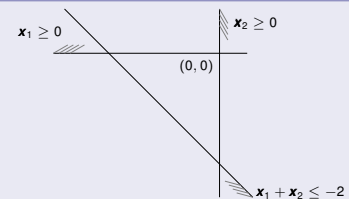
Graphical method: Infeasible problem

Example

Show that the following problem has no feasible solution.

$$\begin{array}{llll} \text{Maximize} & \mathbf{x}_1 & + & \mathbf{x}_2 \\ & \mathbf{x}_1 & + & \mathbf{x}_2 \leq -2 \\ & \mathbf{x}_1, \mathbf{x}_2 & \geq & 0 \end{array}$$

Solution

Linear and affine spaces in \mathbb{R}^n

Definition

A set $L \subseteq \mathbb{R}^n$ is *linear* (also called a linear space) if

- $\mathbf{0} \in L$,
- $\mathbf{x} + \mathbf{y} \in L$ for every $\mathbf{x}, \mathbf{y} \in L$ and
- $\alpha \mathbf{x} \in L$ for every $\mathbf{x} \in L$ and $\alpha \in \mathbb{R}$.

Definition

If $L \subseteq \mathbb{R}^n$ is a linear space and $\mathbf{a} \in \mathbb{R}^n$ is a vector, then $L + \mathbf{a} = \{\mathbf{x} + \mathbf{a}; \mathbf{x} \in L\}$ is called an *affine space*.

Observation

An affine space $A \subseteq \mathbb{R}^n$ is linear if and only if A contains the origin $\mathbf{0}$.

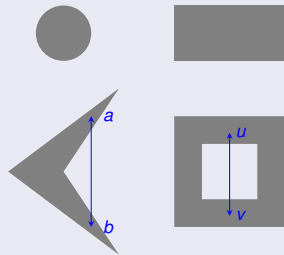
Observation

If $A \subseteq \mathbb{R}^n$ is an affine space, then $A - \mathbf{x}$ is a linear space for every $\mathbf{x} \in A$. Furthermore, all spaces $A - \mathbf{x}$ are the same for all $\mathbf{x} \in A$.

Definition

A set $S \subseteq \mathbb{R}^n$ is *convex* if S contains whole segment between every two points of S .

Example



Linear, affine and convex combinations

Definition

- The sum $\sum_{i=1}^k \alpha_i \mathbf{a}_i$ is called a *linear combination* of $S \subseteq \mathbb{R}^n$ if $k \in \mathbb{N}$, $\mathbf{a}_i \in S$ and $\alpha_i \in \mathbb{R}$ for $i = 1, \dots, k$.
- The sum $\sum_{i=1}^k \alpha_i \mathbf{a}_i$ is called an *affine combination* of $S \subseteq \mathbb{R}^n$ if $k \in \mathbb{N}$, $\mathbf{a}_i \in S$, $\alpha_i \in \mathbb{R}$ and $\sum_{i=1}^k \alpha_i = 1$ for $i = 1, \dots, k$.
- The sum $\sum_{i=1}^k \alpha_i \mathbf{a}_i$ is called a *convex combination* of $S \subseteq \mathbb{R}^n$ if $k \in \mathbb{N}$, $\mathbf{a}_i \in S$, $\alpha_i \geq 0$ and $\sum_{i=1}^k \alpha_i = 1$ for $i = 1, \dots, k$.

Theorem

- The linear hull of a set $S \subseteq \mathbb{R}^n$ is the set of all linear combinations of S .
- The affine hull of a set $S \subseteq \mathbb{R}^n$ is the set of all affine combinations of S .
- The convex hull of a set $S \subseteq \mathbb{R}^n$ is the set of all convex combinations of S .

- Let $\mathbf{a} = \sum \alpha_i \mathbf{a}_i$ and $\mathbf{b} = \sum \beta_j \mathbf{b}_j$ be convex combinations of C . The point $\mathbf{x} = \alpha \mathbf{a} + \beta \mathbf{b}$ on the segment between \mathbf{a} and \mathbf{b} is also a convex combination of C since $\mathbf{x} = \sum \alpha \alpha_i \mathbf{a}_i + \sum \beta \beta_j \mathbf{b}_j$.
- By induction by k , we prove for every $X \subseteq C$ that every convex combinations of k points of X belong into C . Let $\sum \alpha_i \mathbf{a}_i$ be a convex combination of points of X . WLOG $\alpha_j > 0$. For $k = 2$ the statement follows from the definition of convexity. For $k > 2$, let $\alpha' = \alpha_1 + \alpha_2$ and $\mathbf{a}' = \frac{\alpha_1}{\alpha'} \mathbf{a}_1 + \frac{\alpha_2}{\alpha'} \mathbf{a}_2$. Since \mathbf{a}' is a point on the segment between \mathbf{a}_1 and \mathbf{a}_2 it follows that $\mathbf{a}' \in C$. Now, $\alpha' \mathbf{a}' + \sum_{i=3}^k \alpha_i \mathbf{a}_i = \sum \alpha_i \mathbf{a}_i$ is a convex combination of $k - 1$ points of $X \cup \{\mathbf{a}'\}$, so it is contained in C by the induction hypotheses.

Dimension

Observation

Vectors $\mathbf{x}_0, \dots, \mathbf{x}_k$ are affinely independent if and only if vectors $\mathbf{x}_1 - \mathbf{x}_0, \dots, \mathbf{x}_k - \mathbf{x}_0$ are linearly independent.

Observation

Let S be a linear space and $B \subseteq S \setminus \{\mathbf{0}\}$. Then, B is a linear base of S if and only if $B \cup \{\mathbf{0}\}$ is an affine base of S .

Definition

- The *dimension* of a linear space is the cardinality of its linear base.
- The *dimension* of an affine space is the cardinality of its affine base minus one.
- The *dimension* $\dim(S)$ of a set $S \subseteq \mathbb{R}^n$ is the dimension of affine hull of S .

Observation

- A set of vectors S is linearly independent if and only if $\mathbf{0}$ is not a non-trivial linear combination of S .
- A set of vectors S is affinely independent if and only if $\mathbf{0}$ is not a non-trivial combination $\sum \alpha_i \mathbf{a}_i$ of S such that $\sum \alpha_i = 0$ and $\alpha \neq \mathbf{0}$.

Linear, affine and convex hulls

Observation

- The intersection of arbitrary many linear spaces is also a linear space.
- The intersection of arbitrary many affine spaces is also an affine space.
- The intersection of arbitrary many convex sets is also a convex set.

Definition

- The *linear hull* $\text{span}(S)$ of $S \subseteq \mathbb{R}^n$ is the intersection of all linear sets containing S .
- The *affine hull* $\text{aff}(S)$ of $S \subseteq \mathbb{R}^n$ is the intersection of all affine sets containing S .
- The *convex hull* $\text{conv}(S)$ of $S \subseteq \mathbb{R}^n$ is the intersection of all convex sets containing S .

Informally

The linear, the affine and the convex hull of a set $S \subseteq \mathbb{R}^n$ is the smallest (with respect to inclusion) linear, affine and convex set containing S , respectively.

Observation

- A set $S \subseteq \mathbb{R}^n$ is linear if and only if $S = \text{span}(S)$.
- A set $S \subseteq \mathbb{R}^n$ is affine if and only if $S = \text{aff}(S)$.
- A set $S \subseteq \mathbb{R}^n$ is convex if and only if $S = \text{conv}(S)$.

Convex hull and convex combinations

Observation

The set of all convex combinations of a set $C \subseteq \mathbb{R}^n$ is convex. ①

Observation

If $C \subseteq \mathbb{R}^n$ is a convex set and $X \subseteq C$, then C contains all convex combinations of X . ②

Theorem

The convex hull of a set $S \subseteq \mathbb{R}^n$ is the set of all convex combinations of S .

Proof

- Let Z be the set of all convex combinations of S .
- $\text{conv}(S) \subseteq Z$: Observe that Z is a convex set containing S .
- $Z \subseteq \text{conv}(S)$: Observe that convex combinations of points of S belong into $\text{conv}(S)$.

Independence and base

Definition

- A set of vectors $S \subseteq \mathbb{R}^n$ is *linearly independent* if no vector of S is a linear combination of others.
- A set of vectors $S \subseteq \mathbb{R}^n$ is *affinely independent* if no vector of S is an affine combination of others.

Definition

- A set of vectors $B \subseteq \mathbb{R}^n$ is a (*linear*) *base* of a linear space S if vectors of B are linearly independent and $\text{span}(B) = S$.
- A set of vectors $B \subseteq \mathbb{R}^n$ is an (*affine*) *base* of an affine space S if vectors of B are affinely independent and $\text{aff}(B) = S$.

Question

Is it possible to analogously define a convex independence and a convex base?

Observation

- All linear bases of a linear space have the same cardinality.
- All affine bases of an affine space have the same cardinality.

Carathéodory

Theorem (Carathéodory)

Let $S \subseteq \mathbb{R}^n$. Every point of $\text{conv}(S)$ is a convex combinations of affinely independent points of S . ①

Corollary

Let $S \subseteq \mathbb{R}^n$ be a set of dimension d . Then, every point of $\text{conv}(S)$ is a convex combinations of at most $d + 1$ points of S .

- Let $\mathbf{x} \in \text{conv}(S)$. Let $\mathbf{x} = \sum_{i=1}^k \alpha_i \mathbf{x}_i$ be a convex combination of points of S with the smallest k . If $\mathbf{x}_1, \dots, \mathbf{x}_k$ are affinely dependent, then there exists a combination $\mathbf{0} = \sum \beta_i \mathbf{x}_i$ such that $\sum \beta_i = 0$ and $\beta_j \neq 0$. Since this combination is non-trivial, there exists j such that $\beta_j > 0$ and $\frac{\alpha_j}{\beta_j}$ is minimal. Let $\gamma_i = \alpha_i - \frac{\alpha_j \beta_i}{\beta_j}$. Observe that

- $\mathbf{x} = \sum_{i \neq j} \gamma_i \mathbf{x}_i$
- $\sum_{i \neq j} \gamma_i = 1$
- $\gamma_i \geq 0$ for all $i \neq j$

which contradicts the minimality of k .

Mathematical analysis

Definition

- A set $S \subseteq \mathbb{R}^n$ is *closed* if S contains the limit of every converging sequence of points of S .
- A set $S \subseteq \mathbb{R}^n$ is *bounded* if $\max \{ \|\mathbf{x}\|; \mathbf{x} \in S \} < b$ for some $b \in \mathbb{R}$.
- A set $S \subseteq \mathbb{R}^n$ is *compact* if every sequence of points of S contains a converging subsequence with limit in S .

Theorem

A set $S \subseteq \mathbb{R}^n$ is compact if and only if S is closed and bounded.

Theorem

If $f: S \rightarrow \mathbb{R}$ is a continuous function on a compact set $S \subseteq \mathbb{R}^n$, then S contains a point \mathbf{x} maximizing f over S ; that is, $f(\mathbf{x}) \geq f(\mathbf{y})$ for every $\mathbf{y} \in S$.

Infimum and supremum

- Infimum of a set $S \subseteq \mathbb{R}$ is $\inf(S) = \max \{ b \in \mathbb{R}; b \leq x \forall x \in S \}$.
- Supremum of a set $S \subseteq \mathbb{R}$ is $\sup(S) = \min \{ b \in \mathbb{R}; b \geq x \forall x \in S \}$.
- $\inf(\emptyset) = \infty$ and $\sup(\emptyset) = -\infty$
- $\inf(S) = -\infty$ if S has no lower bound

- Find $\mathbf{c} \in C$ and $\mathbf{d} \in D$ with minimal distance $\|\mathbf{d} - \mathbf{c}\|$.
- 1 Let $m = \inf \{ \|\mathbf{d} - \mathbf{c}\|; \mathbf{c} \in C, \mathbf{d} \in D \}$.
 - 2 For every $n \in \mathbb{N}$ there exists $\mathbf{c}_n \in C$ and $\mathbf{d}_n \in D$ such that $\|\mathbf{d}_n - \mathbf{c}_n\| \leq m + \frac{1}{n}$.
 - 3 Since C is compact, there exists a subsequence $\{\mathbf{c}_{n_k}\}_{k=1}^{\infty}$ converging to $\mathbf{c} \in C$.
 - 4 There exists $z \in \mathbb{R}$ such that for every $n \in \mathbb{N}$ the distance $\|\mathbf{d}_n - \mathbf{c}\|$ is at most z : $\|\mathbf{d}_n - \mathbf{c}\| \leq \|\mathbf{d}_n - \mathbf{c}_n\| + \|\mathbf{c}_n - \mathbf{c}\| \leq m + 1 + \max \{ \|\mathbf{c}' - \mathbf{c}''\|; \mathbf{c}', \mathbf{c}'' \in C \} = z$
 - 5 Since the set $D \cap \{ \mathbf{x} \in \mathbb{R}^n; \|\mathbf{x} - \mathbf{c}\| \leq z \}$ is compact, the sequence $\{\mathbf{d}_{n_k}\}_{k=1}^{\infty}$ has a subsequence $\{\mathbf{d}_{n_{k_l}}\}_{l=1}^{\infty}$ converging to $\mathbf{d} \in D$.
 - 6 Since $\|\mathbf{d} - \mathbf{c}\| \leq \|\mathbf{d} - \mathbf{d}_{n_{k_l}}\| + \|\mathbf{d}_{n_{k_l}} - \mathbf{c}_{n_{k_l}}\| + \|\mathbf{c}_{n_{k_l}} - \mathbf{c}\| \rightarrow m$, the distance $\|\mathbf{d} - \mathbf{c}\| = m$ is minimal.
- The required hyperplane is $\mathbf{a}^T \mathbf{x} = b$ where $\mathbf{a} = \mathbf{d} - \mathbf{c}$ and $b = \frac{\mathbf{a}^T \mathbf{c} + \mathbf{a}^T \mathbf{d}}{2}$ since we prove that $\mathbf{a}^T \mathbf{c}' \leq \mathbf{a}^T \mathbf{c} < b < \mathbf{a}^T \mathbf{d} \leq \mathbf{a}^T \mathbf{d}'$ for every $\mathbf{c}' \in C$ and $\mathbf{d}' \in D$.
- 1 In order to prove the most left inequality, let $\mathbf{c}' \in C$.
 - 2 Since C is convex, $\mathbf{y} = \mathbf{c} + \alpha(\mathbf{c}' - \mathbf{c}) \in C$ for every $0 \leq \alpha \leq 1$.
 - 3 From the minimality of the distance $\|\mathbf{d} - \mathbf{c}\|$ it follows that $\|\mathbf{d} - \mathbf{y}\|^2 \geq \|\mathbf{d} - \mathbf{c}\|^2$.

$$\begin{aligned} (\mathbf{d} - \mathbf{c} - \alpha(\mathbf{c}' - \mathbf{c}))^T (\mathbf{d} - \mathbf{c} - \alpha(\mathbf{c}' - \mathbf{c})) &\geq (\mathbf{d} - \mathbf{c})^T (\mathbf{d} - \mathbf{c}) \\ \alpha^2 (\mathbf{c}' - \mathbf{c})^T (\mathbf{c}' - \mathbf{c}) - 2\alpha (\mathbf{d} - \mathbf{c})^T (\mathbf{c}' - \mathbf{c}) &\geq 0 \\ \frac{\alpha}{2} \|\mathbf{c}' - \mathbf{c}\|^2 + \mathbf{a}^T \mathbf{c} &\geq \mathbf{a}^T \mathbf{c}' \end{aligned}$$

- Since the last inequality holds for arbitrarily small $\alpha > 0$, it follows that $\mathbf{a}^T \mathbf{c} \geq \mathbf{a}^T \mathbf{c}'$ holds.

- Clearly, all solutions of $\mathbf{Ax} = \mathbf{0}$ form a linear space S . For every solution \mathbf{z} of $\mathbf{Ax} = \mathbf{b}$ it holds that $S + \mathbf{z}$ is the affine space of all solutions of $\mathbf{Ax} = \mathbf{b}$. Let S be a linear space. Let rows of a matrix A be a linear base of the orthogonal space to S . Then, S are all solutions of $\mathbf{Ax} = \mathbf{0}$. If $S + \mathbf{z}$ is an affine space and $\mathbf{b} = A\mathbf{z}$, then $S + \mathbf{z}$ are all solutions of $\mathbf{Ax} = \mathbf{b}$.

System of linear equations and inequalities

Definition

- A *hyperplane* is a set $\{ \mathbf{x} \in \mathbb{R}^n; \mathbf{a}^T \mathbf{x} = b \}$ where $\mathbf{a} \in \mathbb{R}^n \setminus \{ \mathbf{0} \}$ and $b \in \mathbb{R}$.
- A *half-space* is a set $\{ \mathbf{x} \in \mathbb{R}^n; \mathbf{a}^T \mathbf{x} \leq b \}$ where $\mathbf{a} \in \mathbb{R}^n \setminus \{ \mathbf{0} \}$ and $b \in \mathbb{R}$.
- A *polyhedron* is an intersection of finitely many half-spaces.
- A *polytope* is a bounded polyhedron.

Observation

For every $\mathbf{a} \in \mathbb{R}^n$ and $b \in \mathbb{R}$, the set of all $\mathbf{x} \in \mathbb{R}^n$ satisfying $\mathbf{a}^T \mathbf{x} \leq b$ is convex.

Corollary

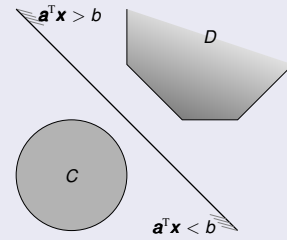
Every polyhedron $\mathbf{Ax} \leq \mathbf{b}$ is convex.

Hyperplane separation theorem

Theorem (strict version)

Let $C, D \subseteq \mathbb{R}^n$ be non-empty, closed, convex and disjoint sets and C be bounded. Then, there exists a hyperplane $\mathbf{a}^T \mathbf{x} = b$ which strictly separates C and D ; that is $C \subseteq \{ \mathbf{x}; \mathbf{a}^T \mathbf{x} < b \}$ and $D \subseteq \{ \mathbf{x}; \mathbf{a}^T \mathbf{x} > b \}$.

Example



Closed convex sets and systems of linear inequalities

Corollary

The intersection of arbitrary many half-spaces is a closed convex set and every closed convex set is an intersection of (infinitely) many half-spaces.

Observation

- The set of all solutions of $\mathbf{Ax} = \mathbf{0}$ is a linear space and every linear space is the set of all solutions of $\mathbf{Ax} = \mathbf{0}$ for some A .
- The set of all solutions of $\mathbf{Ax} = \mathbf{b}$ is an affine space and every affine space is the set of all solutions of $\mathbf{Ax} = \mathbf{b}$ for some A and \mathbf{b} , assuming $\mathbf{Ax} = \mathbf{b}$ is consistent. ①

Definition

The set of all solutions of $\mathbf{Ax} \leq \mathbf{b}$ is called a polyhedron.

Faces of a polyhedron

Definition

Let P be a polyhedron. A half-space $\alpha^T \mathbf{x} \leq \beta$ is called a *supporting hyperplane* of P if the inequality $\alpha^T \mathbf{x} \leq \beta$ holds for every $\mathbf{x} \in P$ and the hyperplane $\alpha^T \mathbf{x} = \beta$ has a non-empty intersection with P .

The set of point in the intersection $P \cap \{ \mathbf{x}; \alpha^T \mathbf{x} = \beta \}$ is called a *face* of P . By convention, the empty set and P are also faces, and the other faces are *proper* faces. ①

Definition

Let P be a d -dimensional polyhedron.

- A 0-dimensional face of P is called a *vertex* of P .
- A 1-dimensional face of P is called an *edge* of P .
- A $(d - 1)$ -dimensional face of P is called a *facet* of P .

Observation

Let $P = \{ \mathbf{x}; \mathbf{Ax} \leq \mathbf{b} \}$ of dimension d . Then for every row i , either

- $P \cap \{ \mathbf{x}; A_{i,*} \mathbf{x} = b_i \} = P$ or
- $P \cap \{ \mathbf{x}; A_{i,*} \mathbf{x} = b_i \} = \emptyset$ or
- $P \cap \{ \mathbf{x}; A_{i,*} \mathbf{x} = b_i \}$ is a proper face of dimension at most $d - 1$.

- Observe, that every face of a polyhedron is also a polyhedron.

⇒ Proof by induction on $d = \dim(S)$:

- For $d = 0$, the size of S is 0 or 1.
- For $d > 0$, let $S = \{x; Ax \leq b\}$ and $S_i = S \cap \{x; A_{i,*}x = b_i\}$. Let I be the set of rows i such that S_i is a proper face of S . Since $\dim(S_i) \leq \dim(S) - 1$ for all $i \in I$, the induction assumption implies that there exists a finite set $V_i \subseteq \mathbb{R}^n$ such that $S_i = \text{conv}(V_i)$. Let $V = \cup_{i \in I} V_i$. We prove that $\text{conv}(V) = S$.
 - follows from $V_i \subseteq S_i \subseteq S$.
 - Let $x \in S$. Let L be a line containing x . $S \cap L$ is a line segment with end-vertices u and v . There exists $i, j \in I$ such that $A_{i,*}u = b_i$ and $A_{j,*}v = b_j$. Since $u \in S_i$ and $v \in S_j$, points u and v are convex combinations of S . Since x is also a convex combination of u and v , we have $x \in \text{conv}(S)$.

- (1) ⇒ (2) Q_1 is the set of all conditions satisfied by all points of $\text{conv}(V)$.
- (1) ⇐ (2) Use the hyperplane separation theorem to separate $x \notin \text{conv}(V)$ from $\text{conv}(V)$.
- (2) ⇐ (3) A condition $\alpha^T v \leq \beta$ is satisfied by all $v \in V$ if and only if the condition is satisfied by $v \in \text{conv}(V)$, so $Q_1 = Q_2$.
- (3) ⇐ (4) α and β in every condition $\alpha^T v \leq \beta$ can be scaled so that $-1 \leq \alpha \leq 1$ and $-1 \leq \beta \leq 1$ and the condition describe the same half-space.
- (4) ⇐ (5) Prove that if $\alpha^T x \leq \beta$ holds for all conditions from W , then it also holds for all conditions from $Q = \text{conv}(W)$.

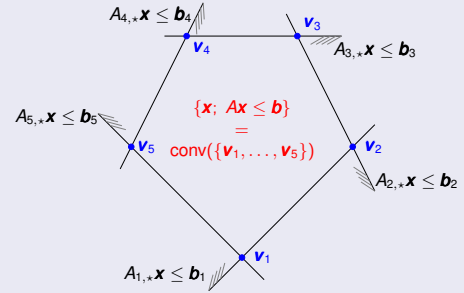
- For simplicity, we prove this theorem only for bounded polyhedrons. Let V_0 be (inclusion) minimal set such that $P = \text{conv}(V_0)$. Let $V_e = \{x \in P; x \notin \text{conv}(P \setminus \{x\})\}$. We prove that $V \subseteq V_e \subseteq V_0 \subseteq V$.
 - $V \subseteq V_e$: Let $z \in V$ be a vertex. By definition, there exists a supporting hyperplane $c^T x = t$ such that $P \cap \{x; c^T x = t\} = \{z\}$. Since $c^T x < t$ for all $x \in P \setminus \{z\}$, it follows that $z \in V_e$.
 - $V_e \subseteq V_0$: Let $z \in V_e$. Since $\text{conv}(P \setminus \{z\}) \neq P$, it follows that $z \in V_0$.
 - $V_0 \subseteq V$: Let $z \in V_0$ and $D = \text{conv}(V_0 \setminus \{z\})$. From Minkovsky-Weil's theorem it follows that V_0 is finite and therefore, D is compact. By the separation theorem, there exists a hyperplane $c^T x = r$ separating $\{z\}$ and D , that is $c^T x < r < c^T z$ for all $x \in D$. Let $t = c^T z$. Hence, $A = \{x; c^T x = t\}$ is a supporting hyperplane of P . We prove that $A \cap P = \{z\}$. For contradiction, let $z' \in P \cap A$ be a different from z . Then, there exists a convex combination $z' = \alpha_1 x_1 + \dots + \alpha_k x_k + \alpha_0 z$ of V_0 . From $z \neq z'$ it follows that $\alpha_0 < 1$ and $\alpha_i > 0$ for some i . Since $\alpha_0 c^T z = t$ and $\alpha_i c^T x_i < t$ and $\alpha_j c^T x_j \leq t$, it holds that $c^T z' < t$ which contradicts the assumption that $z' \in A$.

Minkowski-Weyl

Theorem (Minkowski-Weyl)

A set $S \subseteq \mathbb{R}^n$ is a polytope if and only if there exists a finite set $V \subseteq \mathbb{R}^n$ such that $S = \text{conv}(V)$.

Illustration



Minkowski-Weyl

Theorem (Minkowski-Weyl)

A set $S \subseteq \mathbb{R}^n$ is a polytope if and only if there exists a finite set $V \subseteq \mathbb{R}^n$ such that $S = \text{conv}(V)$.

Proof of the implication ⇐ (main steps)

- Let $Q = \left\{ \begin{pmatrix} \alpha \\ \beta \end{pmatrix}; \alpha \in \mathbb{R}^n, \beta \in \mathbb{R}, -1 \leq \alpha \leq 1, -1 \leq \beta \leq 1, \alpha^T v \leq \beta \forall v \in V \right\}$.
 - Observe that $\alpha^T v \leq \beta$ means the same as $\begin{pmatrix} v \\ -1 \end{pmatrix}^T \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \leq 0$.
 - Since Q is a polytope, there exists a finite set $W \subseteq \mathbb{R}^{n+1}$ such that $Q = \text{conv}(W)$.
 - We prove that $\text{conv}(V) = \left\{ x \in \mathbb{R}^n; \alpha^T x \leq \beta \forall \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in W \right\}$.
- $x \in \text{conv}(V)$
 - $\alpha^T x \leq \beta \forall \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in Q_1$ where $Q_1 = \left\{ \begin{pmatrix} \alpha \\ \beta \end{pmatrix}; \alpha^T v \leq \beta \forall v \in \text{conv}(V) \right\}$
 - $\alpha^T x \leq \beta \forall \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in Q_2$ where $Q_2 = \left\{ \begin{pmatrix} \alpha \\ \beta \end{pmatrix}; \alpha^T v \leq \beta \forall v \in V \right\}$
 - $\alpha^T x \leq \beta \forall \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in Q$
 - $\alpha^T x \leq \beta \forall \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in W$

Faces

Observation

The intersection of two faces of a polyhedron P is a face of P .

Theorem

Let P be a polyhedron and V its vertices. Then, x is a vertex of P if and only if $x \notin \text{conv}(P \setminus \{x\})$. Furthermore, if P is bounded, then $P = \text{conv}(V)$. ①

Observation (A face of a face is a face)

Let F be a face of a polyhedron P and let $E \subseteq F$. Then, E is a face of F if and only if E is a face of P .

Corollary

A set $F \subseteq \mathbb{R}^n$ is a face of a polyhedron $P = \{x \in \mathbb{R}^n; Ax \leq b\}$ if and only if F is the set of all optimal solutions of the linear programming problem $\min \{c^T x; Ax \leq b\}$ for some vector $c \in \mathbb{R}^n$.

Minimal defining system of a polyhedron

Definition

$P = \{x \in \mathbb{R}^n; A'x = b', A''x \leq b''\}$ is a *minimal defining system* of a polyhedron P if

- no condition can be removed and
- no inequality can be replaced by equality without changing the polyhedron P .

Observation

Let z be a point of a polyhedron $P = \{x \in \mathbb{R}^n; A'x = b', A''x \leq b''\}$ such that $A''z < b''$. Then,

- $\dim(P) = n - \text{rank}(A')$ and ①
- z does not belong in any proper face of P . ②

Furthermore, there exists such a point z in every minimal defining system of a polyhedron. ③

Theorem

Let $P = \{x \in \mathbb{R}^n; A'x = b', A''x \leq b''\}$ be a minimal defining system of a polyhedron P . Then, there exists a bijection between facets of P and inequalities $A''x \leq b''$. ④

- Let L be the affine space defined by $A'x = b'$. Clearly, $\dim(P) \leq \dim(L) = n - \text{rank}(A')$. Since $A''z < b''$, there exists $\epsilon > 0$ such that P contains whole ball $B = \{x \in L; \|x - z\| \leq \epsilon\}$. Since vectors of a base of the linear space $L - z$ can be scaled so that they belong into $B - z$, it follows that $\dim(P) \geq \dim(B) \geq \dim(L)$.
- The point z cannot belong into any proper face of P because a supporting hyperplane of such a face split the ball B .
- For every row i of $A''x \leq b''$ there exists $z^i \in P$ such that $A''_{i,*}z^i < b''_i$. Let $z = \frac{1}{m''} \sum_{i=1}^{m''} z^i$ be the center of gravity. Since z is a convex combination of points of P , point z belongs to P . From $A''_{i,*}z^i < b''_i$, it follows that $A''_{i,*}z < b''_i$, and therefore $A''z < b''$.
- Let $R_i = \{x \in \mathbb{R}^n; A''_{i,*}x = b_i\}$ and $F_i = P \cap R_i$. From minimality it follows that R_i is a supporting hyperplane, and therefore, F_i is a face. Likewise in the previous observation, there exists $z \in F_i$ satisfying $A''_{j,*}z < b_j$ for all $j \neq i$ and so $\dim(F_i) = \dim(P) - 1$. Furthermore, $z \notin F_j$ for all $j \neq i$, so $F_i \neq F_j$ for $j \neq i$. For contradiction, let F be an another facet. There exists a facet i such $F \subseteq F_i$, otherwise $z = \frac{1}{m''} \sum_{i=1}^{m''} z^i$ satisfies strictly all condition contradicting the assumption that F is a proper facet. Since $F \neq F_i$, F is a proper face of F_i and so its dimension is at most $\dim(P) - 2$ contradicting the assumption that F is a proper facet.

- Affine space of dimension $n - 1$ is determined by a unique condition.

Minimal defining system of a polyhedron

Theorem

Let $P = \{x \in \mathbb{R}^n; A'x = b', A''x \leq b''\}$ be a minimal defining system of a polyhedron P . Then, there exists a bijection between facets of P and inequalities $A''x \leq b''$.

Definition

A polyhedron $P \subseteq \mathbb{R}^n$ is of full-dimension if $\dim(P) = n$.

Observation

If P is a full-dimensional polyhedron, then P has exactly one minimal defining system up-to multiplying conditions by constants. ☹

Corollary

Every proper face is an intersection of facets.

Outline

- Linear programming
- Linear, affine and convex sets
- Simplex method
- Duality of linear programming
- Integer linear programming
- Matching
- Ellipsoid method
- Vertex Cover
- Matroid

Notation

Notation used in the Simplex method

- Linear programming problem in the equation form is a problem to find $x \in \mathbb{R}^n$ which maximizes $c^T x$ and satisfies $Ax = b$ and $x \geq 0$ where $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$.
- We assume that rows of A are linearly independent.
- For a subset $B \subseteq \{1, \dots, n\}$, let A_B be the matrix consisting of columns of A whose indices belong to B .
- Similarly for vectors, x_B denotes the coordinates of x whose indices belong to B .
- The set $N = \{1, \dots, n\} \setminus B$ denotes the remaining columns.

Example

Consider $B = \{2, 4\}$. Then, $N = \{1, 3, 5\}$ and

$$A = \begin{pmatrix} 1 & 3 & 5 & 6 & 0 \\ 2 & 4 & 8 & 9 & 7 \end{pmatrix} \quad A_B = \begin{pmatrix} 3 & 6 \\ 4 & 9 \end{pmatrix} \quad A_N = \begin{pmatrix} 1 & 5 & 0 \\ 2 & 8 & 7 \end{pmatrix}$$

$$x^T = (3, 4, 6, 2, 7) \quad x_B^T = (4, 2) \quad x_N^T = (3, 6, 7)$$

Note that $Ax = A_B x_B + A_N x_N$.

Example: Initial simplex tableau

Canonical form

$$\begin{array}{rcll} \text{Maximize} & x_1 & + & x_2 \\ & -x_1 & + & x_2 & \leq & 1 \\ & x_1 & & & \leq & 3 \\ & & & x_2 & \leq & 2 \\ & & & x_1, x_2 & \geq & 0 \end{array}$$

Equation form

$$\begin{array}{rcll} \text{Maximize} & x_1 & + & x_2 & & & & & & \\ & -x_1 & + & x_2 & + & x_3 & & & & = & 1 \\ & x_1 & & & + & x_4 & & & & = & 3 \\ & & & x_2 & & & + & x_5 & & = & 2 \\ & & & & & x_1, x_2, x_3, x_4, x_5 & & & & \geq & 0 \end{array}$$

Simplex tableau

$$\begin{array}{rcll} x_3 & = & 1 & + & x_1 & - & x_2 \\ x_4 & = & 3 & - & x_1 & & \\ x_5 & = & 2 & & & - & x_2 \\ \hline z & = & & & x_1 & + & x_2 \end{array}$$

Basic feasible solutions

Definitions

- A set of columns B is a *base* if A_B is a regular matrix.
- The *basic solution* x corresponding to a base B is $x_N = 0$ and $x_B = A_B^{-1}b$.
- A basic solution satisfying $x \geq 0$ is called *basic feasible solution*.

Observation

Basic feasible solutions are exactly vertices of the polyhedron $P = \{x; Ax = b, x \geq 0\}$.

Lemma

A feasible solution x is basic if and only if the columns of the matrix A_K are linearly independent where $K = \{j \in \{1, \dots, n\}; x_j > 0\}$.

Example: Initial simplex tableau

Simplex tableau

$$\begin{array}{rcll} x_3 & = & 1 & + & x_1 & - & x_2 \\ x_4 & = & 3 & - & x_1 & & \\ x_5 & = & 2 & & & - & x_2 \\ \hline z & = & & & x_1 & + & x_2 \end{array}$$

Initial basic feasible solution

- $B = \{3, 4, 5\}$, $N = \{1, 2\}$
- $x = (0, 0, 1, 3, 2)$

Pivot

Two edges from the vertex $(0, 0, 1, 3, 2)$:

- $(t, 0, 1 + t, 3 - t, 2)$ when x_1 is increased by t
- $(0, r, 1 - r, 3, 2 - r)$ when x_2 is increased by r

These edges give feasible solutions for:

- $t \leq 3$ since $x_3 = 1 + t \geq 0$ and $x_4 = 3 - t \geq 0$ and $x_5 = 2 \geq 0$
- $r \leq 1$ since $x_3 = 1 - r \geq 0$ and $x_4 = 3 \geq 0$ and $x_5 = 2 - r \geq 0$

In both cases, the objective function is increasing. We choose x_2 as a pivot.

Example: Pivot step

Simplex tableau

$$\begin{array}{rclcl} x_3 & = & 1 & + & x_1 & - & x_2 \\ x_4 & = & 3 & - & x_1 & & \\ x_5 & = & 2 & & & - & x_2 \\ \hline z & = & & & x_1 & + & x_2 \end{array}$$

Basis

- Original basis $B = \{3, 4, 5\}$
- x_2 enters the basis (by our choice).
- $(0, r, 1 - r, 3, 2 - r)$ is feasible for $r \leq 1$ since $x_3 = 1 - r \geq 0$.
- Therefore, x_3 leaves the basis.
- New base $B = \{2, 4, 5\}$

New simplex tableau

$$\begin{array}{rclcl} x_2 & = & 1 & + & x_1 & - & x_3 \\ x_4 & = & 3 & - & x_1 & & \\ x_5 & = & 1 & - & x_1 & + & x_3 \\ \hline z & = & 1 & + & 2x_1 & - & x_3 \end{array}$$

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Example: Last step

Simplex tableau

$$\begin{array}{rclcl} x_1 & = & 1 & + & x_3 & - & x_5 \\ x_2 & = & 2 & & & - & x_5 \\ x_4 & = & 2 & - & x_3 & + & x_5 \\ \hline z & = & 3 & + & x_3 & - & 2x_5 \end{array}$$

Next pivot

- Basis $B = \{1, 2, 4\}$ with a basis feasible solution $(1, 2, 0, 2, 0)$.
- This vertex has two incident edges but only one increases the objective function.
- The edge increasing objective function is $(1 + t, 2, t, 2 - t, 0)$.
- Feasible solutions for $x_1 = 1 + t \geq 0$ and $x_2 = 2 \geq 0$ and $x_4 = 2 - t \geq 0$.
- Therefore, x_3 enters the basis and x_4 leaves the basis.

New simplex tableau

$$\begin{array}{rclcl} x_1 & = & 3 & - & x_4 \\ x_2 & = & 2 & & - & x_5 \\ x_3 & = & 2 & - & x_4 & + & x_5 \\ \hline z & = & 5 & - & x_4 & - & x_5 \end{array}$$

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Simplex tableau in general

Definition

A simplex tableau determined by a feasible basis B is a system of $m + 1$ linear equations in variables x_1, \dots, x_n , and z that has the same set of solutions as the system $Ax = b$, $z = c^T x$, and in matrix notation looks as follows:

$$\begin{array}{rclcl} x_B & = & p & + & Qx_N \\ \hline z & = & z_0 & + & r^T x_N \end{array}$$

where x_B is the vector of the basis variables, x_N is the vector on non-basis variables, $p \in \mathbb{R}^m$, $r \in \mathbb{R}^{n-m}$, Q is an $m \times (n - m)$ matrix, and $z_0 \in \mathbb{R}$.

Observation

For each basis B there exists exactly one simplex tableau, and it is given by

- $Q = -A_B^{-1} A_N$
- $p = A_B^{-1} b$
- $z_0 = c_B^T A_B^{-1} b$
- $r = c_N^T - (c_B^T A_B^{-1} A_N)^T$

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Pivot step

Simplex tableau in general

$$\begin{array}{rclcl} x_B & = & p & + & Qx_N \\ \hline z & = & z_0 & + & r^T x_N \end{array}$$

Find a pivot

- If $r \leq 0$, then we have an optimal solution.
- Otherwise, choose an arbitrary entering variable x_v such that $r_v > 0$.
- If $Q_{\cdot, v} \geq 0$, then the corresponding edge is unbounded and the problem is also unbounded.
- Otherwise, find a leaving variable x_u which limits the increment of the entering variable most strictly, i.e. $Q_{u, v} < 0$ and $-\frac{p_u}{Q_{u, v}}$ is minimal.

Update the simplex tableau

Gaussian elimination. Postponed for a tutorial.

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Example: Next step

Simplex tableau

$$\begin{array}{rclcl} x_2 & = & 1 & + & x_1 & - & x_3 \\ x_4 & = & 3 & - & x_1 & & \\ x_5 & = & 1 & - & x_1 & + & x_3 \\ \hline z & = & 1 & + & 2x_1 & - & x_3 \end{array}$$

Next pivot

- Basis $B = \{2, 4, 5\}$ with a basis feasible solution $(0, 1, 0, 3, 1)$.
- This vertex has two incident edges but only one increases the objective function.
- The edge increasing objective function is $(t, 1 + t, 0, 3 - t, 1 - t)$.
- Feasible solutions for $x_2 = 1 + t \geq 0$ and $x_4 = 3 - t \geq 0$ and $x_5 = 1 - t \geq 0$.
- Therefore, x_1 enters the basis and x_5 leaves the basis.

New simplex tableau

$$\begin{array}{rclcl} x_1 & = & 1 & + & x_3 & - & x_5 \\ x_2 & = & 2 & & & - & x_5 \\ x_4 & = & 2 & - & x_3 & + & x_5 \\ \hline z & = & 3 & + & x_3 & - & 2x_5 \end{array}$$

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Example: Optimal solution

Simplex tableau

$$\begin{array}{rclcl} x_1 & = & 3 & - & x_4 \\ x_2 & = & 2 & & - & x_5 \\ x_3 & = & 2 & - & x_4 & + & x_5 \\ \hline z & = & 5 & - & x_4 & - & x_5 \end{array}$$

No other pivot

- Basis $B = \{1, 2, 3\}$ with a basis feasible solution $(3, 2, 2, 0, 0)$.
- This vertex has two incident edges but no one increases the objective function.
- We have an optimal solution.

Why this is an optimal solution?

- Consider an arbitrary feasible solution \tilde{y} .
- The value of objective function is $\tilde{z} = 5 - \tilde{y}_4 - \tilde{y}_5$.
- Since $\tilde{y}_4, \tilde{y}_5 \geq 0$, the objective value is $\tilde{z} = 5 - \tilde{y}_4 - \tilde{y}_5 \leq 5 = z$.

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Properties of a simplex tableau

Simplex tableau in general

$$\begin{array}{rclcl} x_B & = & p & + & Qx_N \\ \hline z & = & z_0 & + & r^T x_N \end{array}$$

Observation

Basis B is feasible if and only if $p \geq 0$.

Observation

The solution corresponding to a basis B is optimal if and only if $r \leq 0$.

Observation

If a linear programming problem in the equation form is feasible and bounded, then it has an optimal basis solution.

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Pivot rules

Pivot rules

Largest coefficient Choose an improving variable with the largest coefficient.

Largest increase Choose an improving variable that leads to the largest absolute improvement in z .

Steepest edge Choose an improving variable whose entering into the basis moves the current basic feasible solution in a direction closest to the direction of the vector c , i.e.

$$\frac{c^T(x_{new} - x_{old})}{\|x_{new} - x_{old}\|}$$

Bland's rule Choose an improving variable with the smallest index, and if there are several possibilities of the leaving variable, also take the one with the smallest index.

Random edge Select the entering variable uniformly at random among all improving variables.

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Equation form

Maximize $\mathbf{c}^T \mathbf{x}$ such that $\mathbf{Ax} = \mathbf{b}$ and $\mathbf{x} \geq 0$.

Auxiliary linear program

We introduce variables $\mathbf{x}_{n+1}, \dots, \mathbf{x}_{n+m}$ and solve an auxiliary linear program:
Maximize $-\mathbf{x}_{n+1} \cdots -\mathbf{x}_{n+m}$ such that $(\mathbf{A}|\mathbf{I})\mathbf{x} = \mathbf{b}$ and $\mathbf{x} \geq 0$.

Observation

The original linear program has a feasible solution if and only if an optimal solution of the auxiliary linear program satisfies $\mathbf{x}_{n+1} = \dots = \mathbf{x}_{n+m} = 0$.

- For example, the apex of the 3-dimensional k -side pyramid belongs to k faces, so there are $\binom{k}{3}$ basis determining the apex.
- In degeneracy, the simplex method stay in the same vertex; and when the vertex is left, it is not visited again.
- The Klee-Minty cube is a "deformed" n -dimensional cube with $2n$ facets and 2^n vertices. The Dantzig's original pivot rule (largest coefficient) visits all vertices of this cube.

Duality of linear programming: Example

Find an upper bound for the following problem

$$\begin{array}{llll} \text{Maximize} & 2\mathbf{x}_1 & + & 3\mathbf{x}_2 \\ \text{subject to} & 4\mathbf{x}_1 & + & 8\mathbf{x}_2 \leq 12 \\ & 2\mathbf{x}_1 & + & \mathbf{x}_2 \leq 3 \\ & 3\mathbf{x}_1 & + & 2\mathbf{x}_2 \leq 4 \\ & \mathbf{x}_1, \mathbf{x}_2 & \geq & 0 \end{array}$$

Simple estimates

- $2\mathbf{x}_1 + 3\mathbf{x}_2 \leq 4\mathbf{x}_1 + 8\mathbf{x}_2 \leq 12$ ①
- $2\mathbf{x}_1 + 3\mathbf{x}_2 \leq \frac{1}{2}(4\mathbf{x}_1 + 8\mathbf{x}_2) \leq 6$ ②
- $2\mathbf{x}_1 + 3\mathbf{x}_2 = \frac{1}{3}(4\mathbf{x}_1 + 8\mathbf{x}_2 + 2\mathbf{x}_1 + \mathbf{x}_2) \leq 5$ ③

What is the best combination of conditions?

Every non-negative linear combination of inequalities which gives an inequality $\mathbf{d}_1\mathbf{x}_1 + \mathbf{d}_2\mathbf{x}_2 \leq h$ with $\mathbf{d}_1 \geq 2$ and $\mathbf{d}_2 \geq 3$ provides the upper bound $2\mathbf{x}_1 + 3\mathbf{x}_2 \leq \mathbf{d}_1\mathbf{x}_1 + \mathbf{d}_2\mathbf{x}_2 \leq h$.

Duality of linear programming: Example

Find an upper bound for the following problem

$$\begin{array}{llll} \text{Maximize} & 2\mathbf{x}_1 & + & 3\mathbf{x}_2 \\ \text{subject to} & 4\mathbf{x}_1 & + & 8\mathbf{x}_2 \leq 12 \\ & 2\mathbf{x}_1 & + & \mathbf{x}_2 \leq 3 \\ & 3\mathbf{x}_1 & + & 2\mathbf{x}_2 \leq 4 \\ & \mathbf{x}_1, \mathbf{x}_2 & \geq & 0 \end{array}$$

Non-negative combination of inequalities with coefficients $\mathbf{y}_1, \mathbf{y}_2$ and \mathbf{y}_3

$$(4\mathbf{y}_1 + 2\mathbf{y}_2 + 3\mathbf{y}_3)\mathbf{x}_1 + (8\mathbf{y}_1 + \mathbf{y}_2 + 2\mathbf{y}_3)\mathbf{x}_2 \leq 12\mathbf{y}_1 + 3\mathbf{y}_2 + 4\mathbf{y}_3 \text{ where}$$

- $\mathbf{d}_1 = 4\mathbf{y}_1 + 2\mathbf{y}_2 + 3\mathbf{y}_3 \geq 2$
- $\mathbf{d}_2 = 8\mathbf{y}_1 + \mathbf{y}_2 + 2\mathbf{y}_3 \geq 3$
- $h = 12\mathbf{y}_1 + 3\mathbf{y}_2 + 4\mathbf{y}_3$ to be minimized

Dual program ①

$$\begin{array}{llll} \text{Minimize} & 12\mathbf{y}_1 & + & 3\mathbf{y}_2 & + & 4\mathbf{y}_3 \\ \text{subject to} & 4\mathbf{y}_1 & + & 2\mathbf{y}_2 & + & 3\mathbf{y}_3 \geq 2 \\ & 8\mathbf{y}_1 & + & \mathbf{y}_2 & + & 2\mathbf{y}_3 \geq 3 \\ & \mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3 & \geq & 0 \end{array}$$

Degeneracy

- Different basis may correspond to the same solution. ①
- The simplex method may loop forever between these basis.
- Bland's or lexicographic rules prevent visiting the same basis twice.

The number of visited vertices

- The total number of vertices is finite since the number of basis is finite.
- The objective value of visited vertices is increasing, so every vertex is visited at most once. ②
- The number of visited vertices may be exponential, e.g. the Klee-Minty cube. ③
- Practical linear programming problems in equation forms with m equations typically need between $2m$ and $3m$ pivot steps to solve.

Open problem

Is there a pivot rule which guarantees a polynomial number of steps?

Outline

- Linear programming
- Linear, affine and convex sets
- Simplex method
- Duality of linear programming
- Integer linear programming
- Matching
- Ellipsoid method
- Vertex Cover
- Matroid

- The first condition
- A half of the first condition
- A third of the sum of the first and the second conditions

- The primal optimal solution is $\mathbf{x}^T = (\frac{1}{2}, \frac{5}{4})$ and the dual solution is $\mathbf{y}^T = (\frac{5}{16}, 0, \frac{1}{4})$, both with the same objective value 4.75.

Duality of linear programming: General

Primal linear program

Maximize $\mathbf{c}^T \mathbf{x}$ subject to $\mathbf{Ax} \leq \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$

Dual linear program

Minimize $\mathbf{b}^T \mathbf{y}$ subject to $\mathbf{A}^T \mathbf{y} \geq \mathbf{c}$ and $\mathbf{y} \geq \mathbf{0}$

Weak duality theorem

For every primal feasible solution \mathbf{x} and dual feasible solution \mathbf{y} hold $\mathbf{c}^T \mathbf{x} \leq \mathbf{b}^T \mathbf{y}$.

Corollary

If one program is unbounded, then the other one is infeasible.

Duality theorem

Exactly one of the following possibilities occurs

- Neither primal nor dual has a feasible solution
- Primal is unbounded and dual is infeasible
- Primal is infeasible and dual is unbounded
- There are feasible solutions \mathbf{x} and \mathbf{y} such that $\mathbf{c}^T \mathbf{x} = \mathbf{b}^T \mathbf{y}$

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Dualization: General rules

	Primal linear program	Dual linear program
Variables	$\mathbf{x}_1, \dots, \mathbf{x}_n$	$\mathbf{y}_1, \dots, \mathbf{y}_m$
Matrix	\mathbf{A}	\mathbf{A}^T
Right-hand side	\mathbf{b}	\mathbf{c}
Objective function	max $\mathbf{c}^T \mathbf{x}$	min $\mathbf{b}^T \mathbf{y}$
Constraints	i -the constraint has \leq i -the constraint has \geq i -the constraint has $=$	$\mathbf{y}_i \geq 0$ $\mathbf{y}_i \leq 0$ $\mathbf{y}_i \in \mathbb{R}$
	$\mathbf{x}_j \geq 0$ $\mathbf{x}_j \leq 0$ $\mathbf{x}_j \in \mathbb{R}$	j -th constraint has \geq j -th constraint has \leq j -th constraint has $=$

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Complementary slackness

Theorem

Feasible solutions \mathbf{x} and \mathbf{y} of linear programs

- Primal: Maximize $\mathbf{c}^T \mathbf{x}$ subject to $\mathbf{Ax} \leq \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$
- Dual: Minimize $\mathbf{b}^T \mathbf{y}$ subject to $\mathbf{A}^T \mathbf{y} \geq \mathbf{c}$ and $\mathbf{y} \geq \mathbf{0}$

are optimal if and only if

- $\mathbf{x}_i = 0$ or $\mathbf{A}_{i,*}^T \mathbf{y} = \mathbf{c}_i$ for every $i = 1, \dots, n$ and
- $\mathbf{y}_j = 0$ or $\mathbf{A}_{j,*} \mathbf{x} = \mathbf{b}_j$ for every $j = 1, \dots, m$.

Proof

$$\mathbf{c}^T \mathbf{x} = \sum_{i=1}^n \mathbf{c}_i \mathbf{x}_i \leq \sum_{i=1}^n (\mathbf{y}^T \mathbf{A}_{i,*}) \mathbf{x}_i = \mathbf{y}^T \mathbf{Ax} = \sum_{j=1}^m \mathbf{y}_j (\mathbf{A}_{j,*} \mathbf{x}) \leq \sum_{j=1}^m \mathbf{y}_j \mathbf{b}_j = \mathbf{b}^T \mathbf{y}$$

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Fourier–Motzkin elimination: Example

Rewrite into a system of inequalities

Real numbers x and y satisfy

$$\max \left\{ 7 + 5y - 2z, -4 + \frac{2}{3}y + 2z \right\} \leq \min \left\{ 5 + \frac{5}{2}y - 2z, 3 + 2y - z, 3 - 2y + \frac{1}{5}z \right\}$$

if and only they satisfy

$$\begin{array}{rcl} 7 + 5y - 2z & \leq & 5 + \frac{5}{2}y - 2z \\ 7 + 5y - 2z & \leq & 3 + 2y - z \\ 7 + 5y - 2z & \leq & 3 - 2y + \frac{1}{5}z \\ -4 + \frac{2}{3}y + 2z & \leq & 5 + \frac{5}{2}y - 2z \\ -4 + \frac{2}{3}y + 2z & \leq & 3 + 2y - z \\ -4 + \frac{2}{3}y + 2z & \leq & 3 - 2y + \frac{1}{5}z \end{array}$$

Next steps

Eliminate variables y and z in a similar way.

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Dualization

Every linear programming problem has its dual, e.g.

- Maximize $\mathbf{c}^T \mathbf{x}$ subject to $\mathbf{Ax} \geq \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$
- Maximize $\mathbf{c}^T \mathbf{x}$ subject to $-\mathbf{Ax} \leq -\mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$
- Minimize $-\mathbf{b}^T \mathbf{y}$ subject to $-\mathbf{A}^T \mathbf{y} \geq \mathbf{c}$ and $\mathbf{y} \geq \mathbf{0}$
- Minimize $\mathbf{b}^T \mathbf{y}$ subject to $\mathbf{A}^T \mathbf{y} \geq \mathbf{c}$ and $\mathbf{y} \leq \mathbf{0}$

A dual of a dual problem is the (original) primal problem

- Minimize $\mathbf{b}^T \mathbf{y}$ subject to $\mathbf{A}^T \mathbf{y} \geq \mathbf{c}$ and $\mathbf{y} \geq \mathbf{0}$
- Maximize $-\mathbf{b}^T \mathbf{y}$ subject to $\mathbf{A}^T \mathbf{y} \geq \mathbf{c}$ and $\mathbf{y} \geq \mathbf{0}$
- Minimize $\mathbf{c}^T \mathbf{x}$ subject to $\mathbf{Ax} \geq -\mathbf{b}$ and $\mathbf{x} \leq \mathbf{0}$
- Minimize $-\mathbf{c}^T \mathbf{x}$ subject to $-\mathbf{Ax} \geq -\mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$
- Maximize $\mathbf{c}^T \mathbf{x}$ subject to $\mathbf{Ax} \leq \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$

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Linear programming: Feasibility versus optimality

Feasibility versus optimality

Finding a feasible solution of a linear program is computationally as difficult as finding an optimal solution.

Using duality

The optimal solutions of linear programs

- Primal: Maximize $\mathbf{c}^T \mathbf{x}$ subject to $\mathbf{Ax} \leq \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$
- Dual: Minimize $\mathbf{b}^T \mathbf{y}$ subject to $\mathbf{A}^T \mathbf{y} \geq \mathbf{c}$ and $\mathbf{y} \geq \mathbf{0}$

are exactly feasible solutions satisfying

$$\begin{array}{rcl} \mathbf{Ax} & \leq & \mathbf{b} \\ \mathbf{A}^T \mathbf{y} & \geq & \mathbf{c} \\ \mathbf{c}^T \mathbf{x} & \geq & \mathbf{b}^T \mathbf{y} \\ \mathbf{x}, \mathbf{y} & \geq & \mathbf{0} \end{array}$$

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Fourier–Motzkin elimination: Example

Goal: Find a feasible solution

$$\begin{array}{rcl} 2x & - & 5y & + & 4z & \leq & 10 \\ 3x & - & 6y & + & 3z & \leq & 9 \\ 5x & + & 10y & - & z & \leq & 15 \\ -x & + & 5y & - & 2z & \leq & -7 \\ -3x & + & 2y & + & 6z & \leq & 12 \end{array}$$

Express the variable x in each condition

$$\begin{array}{rcl} x & \leq & 5 & + & \frac{5}{2}y & - & 2z \\ x & \leq & 3 & + & 2y & - & z \\ x & \leq & 3 & - & 2y & + & \frac{1}{5}z \\ x & \geq & 7 & + & 5y & - & 2z \\ x & \geq & -4 & + & \frac{2}{3}y & + & 2z \end{array}$$

Eliminate the variable x

The original system has a feasible solution if and only if there exist y and z satisfying

$$\max \left\{ 7 + 5y - 2z, -4 + \frac{2}{3}y + 2z \right\} \leq \min \left\{ 5 + \frac{5}{2}y - 2z, 3 + 2y - z, 3 - 2y + \frac{1}{5}z \right\}$$

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Fourier–Motzkin elimination: In general

Observation

Let $\mathbf{Ax} \leq \mathbf{b}$ be a system with $n \geq 1$ variables and m inequalities. There is a system $\mathbf{A}'\mathbf{x}' \leq \mathbf{b}'$ with $n - 1$ variables and at most $\max \{m, m^2/4\}$ inequalities, with the following properties:

- $\mathbf{Ax} \leq \mathbf{b}$ has a solution if and only if $\mathbf{A}'\mathbf{x}' \leq \mathbf{b}'$ has a solution, and
- each inequality of $\mathbf{A}'\mathbf{x}' \leq \mathbf{b}'$ is a positive linear combination of some inequalities from $\mathbf{Ax} \leq \mathbf{b}$.

Proof

- WLOG: $\mathbf{A}_{i,1} \in \{-1, 0, 1\}$ for all $i = 1, \dots, n$
- Let $C = \{i; \mathbf{A}_{i,1} = 1\}$, $F = \{i; \mathbf{A}_{i,1} = -1\}$ and $L = \{i; \mathbf{A}_{i,1} = 0\}$
- Let $\mathbf{A}'\mathbf{x}' \leq \mathbf{b}'$ be the system of $n - 1$ variables and $|C| \cdot |F| + |L|$ inequalities

$$\begin{array}{l} j \in C, k \in F: (\mathbf{A}_{j,*} + \mathbf{A}_{k,*})\mathbf{x} \leq \mathbf{b}_j + \mathbf{b}_k \quad (1) \\ l \in L: \mathbf{A}_{l,*}\mathbf{x} \leq \mathbf{b}_l \quad (2) \end{array}$$
- Assuming $\mathbf{A}'\mathbf{x}' \leq \mathbf{b}'$ has a solution \mathbf{x}' , we find a solution \mathbf{x} of $\mathbf{Ax} \leq \mathbf{b}$:
 - (1) is equivalent to $\mathbf{A}_{k,*}'\mathbf{x}' - \mathbf{b}_k \leq \mathbf{b}_j - \mathbf{A}_{j,*}'\mathbf{x}'$ for all $j \in C, k \in F$,
 - which is equivalent to $\max_{k \in F} \{\mathbf{A}_{k,*}'\mathbf{x}' - \mathbf{b}_k\} \leq \min_{j \in C} \{\mathbf{b}_j - \mathbf{A}_{j,*}'\mathbf{x}'\}$
 - Choose \mathbf{x}_1 between these bounds and $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}')$ satisfies $\mathbf{Ax} \leq \mathbf{b}$

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Definition

A cone generated by vectors $\mathbf{a}_1, \dots, \mathbf{a}_n \in \mathbb{R}^m$ is the set of all non-negative combinations of $\mathbf{a}_1, \dots, \mathbf{a}_n$, i.e. $\{\sum_{i=1}^n \alpha_i \mathbf{a}_i; \alpha_1, \dots, \alpha_n \geq 0\}$.

Proposition (Farkas lemma geometrically)

Let $\mathbf{a}_1, \dots, \mathbf{a}_n, \mathbf{b} \in \mathbb{R}^m$. Then exactly one of the following two possibilities occurs:

- 1 The point \mathbf{b} lies in the cone generated by $\mathbf{a}_1, \dots, \mathbf{a}_n$.
- 2 There exists a hyperplane $h = \{\mathbf{x} \in \mathbb{R}^m; \mathbf{y}^T \mathbf{x} = 0\}$ containing $\mathbf{0}$ for some $\mathbf{y} \in \mathbb{R}^m$ separating $\mathbf{a}_1, \dots, \mathbf{a}_n$ and \mathbf{b} , i.e. $\mathbf{y}^T \mathbf{a}_i \geq 0$ for all $i = 1, \dots, n$ and $\mathbf{y}^T \mathbf{b} < 0$.

Proposition (Farkas lemma)

Let $A \in \mathbb{R}^{m \times n}$ and $\mathbf{b} \in \mathbb{R}^m$. Then exactly one of the following two possibilities occurs:

- 1 There exists a vector $\mathbf{x} \in \mathbb{R}^n$ satisfying $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$.
- 2 There exists a vector $\mathbf{y} \in \mathbb{R}^m$ satisfying $\mathbf{y}^T A \geq \mathbf{0}$ and $\mathbf{y}^T \mathbf{b} < 0$.

Farkas lemma

Proposition (Farkas lemma, 3rd version)

Let $A \in \mathbb{R}^{m \times n}$ and $\mathbf{b} \in \mathbb{R}^m$. Then, the system $A\mathbf{x} \leq \mathbf{b}$ has a solution $\mathbf{x} \in \mathbb{R}^n$ if and only if every non-negative $\mathbf{y} \in \mathbb{R}^m$ with $\mathbf{y}^T A = \mathbf{0}^T$ satisfies $\mathbf{y}^T \mathbf{b} \geq 0$.

Proof

- \Rightarrow If \mathbf{x} satisfies $A\mathbf{x} \leq \mathbf{b}$ and $\mathbf{y} \geq \mathbf{0}$ satisfies $\mathbf{y}^T A = \mathbf{0}^T$, then $\mathbf{y}^T \mathbf{b} \geq \mathbf{y}^T A\mathbf{x} \geq \mathbf{0}^T \mathbf{x} = 0$
- \Leftarrow If $A\mathbf{x} \leq \mathbf{b}$ has no solution, then find $\mathbf{y} \geq \mathbf{0}$, $\mathbf{y}^T A = \mathbf{0}^T$, $\mathbf{y}^T \mathbf{b} < 0$ by the induction on n
- $n = 0$
- The system $A\mathbf{x} \leq \mathbf{b}$ equals to $\mathbf{0} \leq \mathbf{b}$ which is infeasible, so $b_i < 0$ for some i
 - Choose $\mathbf{y} = \mathbf{e}_i$ (the i -th unit vector)
- $n > 0$
- Using Fourier–Motzkin elimination we obtain an infeasible system $A'\mathbf{x}' \leq \mathbf{b}'$
 - There exists a non-negative matrix M such that $(\mathbf{0}|A') = MA$ and $\mathbf{b}' = M\mathbf{b}$
 - By induction, there exists $\mathbf{y}' \geq \mathbf{0}$, $\mathbf{y}'^T A' = \mathbf{0}^T$, $\mathbf{y}'^T \mathbf{b}' < 0$
 - We verify that $\mathbf{y} = M^T \mathbf{y}'$ satisfies all requirements of the induction
- $$\mathbf{y} = M^T \mathbf{y}' \geq \mathbf{0}$$
- $$\mathbf{y}^T A = (M^T \mathbf{y}')^T A = \mathbf{y}'^T M A = \mathbf{y}'^T (\mathbf{0}|A') = \mathbf{0}^T$$
- $$\mathbf{y}^T \mathbf{b} = (M^T \mathbf{y}')^T \mathbf{b} = \mathbf{y}'^T M \mathbf{b} = \mathbf{y}'^T \mathbf{b}' < 0^T$$

Proof of the duality of linear programming

Proposition (Farkas lemma, 2nd version)

Let $A \in \mathbb{R}^{m \times n}$ and $\mathbf{b} \in \mathbb{R}^m$. The system $A\mathbf{x} \leq \mathbf{b}$ has a non-negative solution if and only if every non-negative $\mathbf{y} \in \mathbb{R}^m$ with $\mathbf{y}^T A \geq \mathbf{0}^T$ satisfies $\mathbf{y}^T \mathbf{b} \geq 0$.

Duality

- Primal: Maximize $\mathbf{c}^T \mathbf{x}$ subject to $A\mathbf{x} \leq \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$
- Dual: Minimize $\mathbf{b}^T \mathbf{y}$ subject to $A^T \mathbf{y} \geq \mathbf{c}$ and $\mathbf{y} \geq \mathbf{0}$

If the primal problem has an optimal solution \mathbf{x}^* , then the dual problem has an optimal solution \mathbf{y}^* and $\mathbf{c}^T \mathbf{x}^* = \mathbf{b}^T \mathbf{y}^*$.

Proof of duality using Farkas lemma

- 1 Let \mathbf{x}^* be an optimal solution of the primal problem and $\gamma = \mathbf{c}^T \mathbf{x}^*$
- 2 $\epsilon > 0$ iff $A\mathbf{x} \leq \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$ and $\mathbf{c}^T \mathbf{x} \geq \gamma + \epsilon$ is infeasible
- 3 $\epsilon > 0$ iff $(\begin{smallmatrix} A \\ -\mathbf{c}^T \end{smallmatrix}) \mathbf{x} \leq (\begin{smallmatrix} \mathbf{b} \\ -\gamma - \epsilon \end{smallmatrix})$ and $\mathbf{x} \geq \mathbf{0}$ is infeasible
- 4 $\epsilon > 0$ iff $\mathbf{u}, \mathbf{z} \geq \mathbf{0}$ and $(\begin{smallmatrix} \mathbf{u} \\ \mathbf{z} \end{smallmatrix})^T (\begin{smallmatrix} A \\ -\mathbf{c}^T \end{smallmatrix}) \geq \mathbf{0}^T$ and $(\begin{smallmatrix} \mathbf{u} \\ \mathbf{z} \end{smallmatrix})^T (\begin{smallmatrix} \mathbf{b} \\ -\gamma - \epsilon \end{smallmatrix}) < 0$ is feasible
- 5 $\epsilon > 0$ iff $\mathbf{u}, \mathbf{z} \geq \mathbf{0}$ and $A^T \mathbf{u} \geq \mathbf{z} \mathbf{c}$ and $\mathbf{b}^T \mathbf{u} < \mathbf{z}(\gamma + \epsilon)$ is feasible

Outline

- 1 Linear programming
- 2 Linear, affine and convex sets
- 3 Simplex method
- 4 Duality of linear programming
- 5 Integer linear programming
- 6 Matching
- 7 Ellipsoid method
- 8 Vertex Cover
- 9 Matroid

Proposition (Farkas lemma)

Let $A \in \mathbb{R}^{m \times n}$ and $\mathbf{b} \in \mathbb{R}^m$. The following statements hold.

- 1 The system $A\mathbf{x} = \mathbf{b}$ has a non-negative solution $\mathbf{x} \in \mathbb{R}^n$ if and only if every $\mathbf{y} \in \mathbb{R}^m$ with $\mathbf{y}^T A \geq \mathbf{0}^T$ satisfies $\mathbf{y}^T \mathbf{b} \geq 0$.
- 2 The system $A\mathbf{x} \leq \mathbf{b}$ has a non-negative solution $\mathbf{x} \in \mathbb{R}^n$ if and only if every non-negative $\mathbf{y} \in \mathbb{R}^m$ with $\mathbf{y}^T A \geq \mathbf{0}^T$ satisfies $\mathbf{y}^T \mathbf{b} \geq 0$.
- 3 The system $A\mathbf{x} \leq \mathbf{b}$ has a solution $\mathbf{x} \in \mathbb{R}^n$ if and only if every non-negative $\mathbf{y} \in \mathbb{R}^m$ with $\mathbf{y}^T A = \mathbf{0}^T$ satisfies $\mathbf{y}^T \mathbf{b} \geq 0$.

Overview of the proof of duality

Fourier–Motzkin elimination
 \Downarrow
 Farkas lemma, 3rd version
 \Downarrow
 Farkas lemma, 2nd version
 \Downarrow
 Duality of linear programming

Farkas lemma

Proposition (Farkas lemma, 3rd version)

Let $A \in \mathbb{R}^{m \times n}$ and $\mathbf{b} \in \mathbb{R}^m$. Then, the system $A\mathbf{x} \leq \mathbf{b}$ has a solution $\mathbf{x} \in \mathbb{R}^n$ if and only if every non-negative $\mathbf{y} \in \mathbb{R}^m$ with $\mathbf{y}^T A = \mathbf{0}^T$ satisfies $\mathbf{y}^T \mathbf{b} \geq 0$.

Proposition (Farkas lemma, 2nd version)

Let $A \in \mathbb{R}^{m \times n}$ and $\mathbf{b} \in \mathbb{R}^m$. The system $A\mathbf{x} \leq \mathbf{b}$ has a non-negative solution $\mathbf{x} \in \mathbb{R}^n$ if and only if every non-negative $\mathbf{y} \in \mathbb{R}^m$ with $\mathbf{y}^T A \geq \mathbf{0}^T$ satisfies $\mathbf{y}^T \mathbf{b} \geq 0$.

Proof of the 2nd version using the 3rd version

The following statements are equivalent

- 1 $A\mathbf{x} \leq \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$ has a solution
- 2 $(\begin{smallmatrix} A \\ -\mathbf{c}^T \end{smallmatrix}) \mathbf{x} \leq (\begin{smallmatrix} \mathbf{b} \\ -\gamma \end{smallmatrix})$ has a solution
- 3 Every $\mathbf{y}, \mathbf{y}' \geq \mathbf{0}$ with $(\begin{smallmatrix} \mathbf{y} \\ \mathbf{y}' \end{smallmatrix})^T (\begin{smallmatrix} A \\ -\mathbf{c}^T \end{smallmatrix}) = \mathbf{0}^T$ satisfies $(\begin{smallmatrix} \mathbf{y} \\ \mathbf{y}' \end{smallmatrix})^T (\begin{smallmatrix} \mathbf{b} \\ -\gamma \end{smallmatrix}) \geq 0$
- 4 Every $\mathbf{y}, \mathbf{y}' \geq \mathbf{0}$ with $\mathbf{y}^T A = \mathbf{y}'^T \mathbf{c}$ satisfies $\mathbf{y}^T \mathbf{b} \geq \gamma$
- 5 Every $\mathbf{y} \geq \mathbf{0}$ with $\mathbf{y}^T A \geq \mathbf{0}^T$ satisfies $\mathbf{y}^T \mathbf{b} \geq 0$

Proof of the duality of linear programming

Duality

- Primal: Maximize $\mathbf{c}^T \mathbf{x}$ subject to $A\mathbf{x} \leq \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$
- Dual: Minimize $\mathbf{b}^T \mathbf{y}$ subject to $A^T \mathbf{y} \geq \mathbf{c}$ and $\mathbf{y} \geq \mathbf{0}$

If the primal problem has an optimal solution \mathbf{x}^* , then the dual problem has an optimal solution \mathbf{y}^* and $\mathbf{c}^T \mathbf{x}^* = \mathbf{b}^T \mathbf{y}^*$.

Proof of duality using Farkas lemma (continue)

- 1 Let \mathbf{x}^* be an optimal solution of the primal problem and $\gamma = \mathbf{c}^T \mathbf{x}^*$
- 2 $\epsilon > 0$ iff $\mathbf{u}, \mathbf{z} \geq \mathbf{0}$ and $A^T \mathbf{u} \geq \mathbf{z} \mathbf{c}$ and $\mathbf{b}^T \mathbf{u} < \mathbf{z}(\gamma + \epsilon)$ is feasible
- 3 For $\epsilon > 0$, there exists $\mathbf{u}', \mathbf{z}' \geq \mathbf{0}$ with $A^T \mathbf{u}' \geq \mathbf{z}' \mathbf{c}$ and $\mathbf{b}^T \mathbf{u}' < \mathbf{z}'(\gamma + \epsilon)$
- 4 For $\epsilon = 0$ it holds that $\mathbf{u}', \mathbf{z}' \geq \mathbf{0}$ and $A^T \mathbf{u}' \geq \mathbf{z}' \mathbf{c}$ so $\mathbf{b}^T \mathbf{u}' \geq \mathbf{z}' \gamma$
- 5 Since $\mathbf{z}' \gamma \leq \mathbf{b}^T \mathbf{u}' < \mathbf{z}'(\gamma + \epsilon)$ and $\mathbf{z}' \geq \mathbf{0}$ it follows that $\mathbf{z}' > 0$
- 6 Let $\mathbf{v} = \frac{1}{\mathbf{z}'} \mathbf{u}'$
- 7 Since $A^T \mathbf{v} \geq \mathbf{c}$ and $\mathbf{v} \geq \mathbf{0}$, the dual solution \mathbf{v} is feasible
- 8 Since the dual is feasible and bounded, there exists an optimal dual solution \mathbf{y}^*
- 9 Hence, $\mathbf{b}^T \mathbf{y}^* < \gamma + \epsilon$ for every $\epsilon > 0$, and so $\mathbf{b}^T \mathbf{y}^* \leq \gamma$
- 10 From the weak duality theorem it follows that $\mathbf{b}^T \mathbf{y}^* = \mathbf{c}^T \mathbf{x}^*$

Integer linear programming

Integer linear programming

Integer linear programming problem is an optimization problem to find $\mathbf{x} \in \mathbb{Z}^n$ which maximizes $\mathbf{c}^T \mathbf{x}$ and satisfies $A\mathbf{x} \leq \mathbf{b}$ where $A \in \mathbb{R}^{m \times n}$ and $\mathbf{b} \in \mathbb{R}^m$.

Mix integer linear programming

Some variables are integer and others are real.

Relaxed problem and solution

- Given a (mix) integer linear programming problem, the corresponding relaxed problem is the linear programming problem where all integral constraints $\mathbf{x}_i \in \mathbb{Z}$ are relaxed; that is, replaced by $\mathbf{x}_i \in \mathbb{R}$.
- Relaxed solution is a feasible solution of the relaxed problem.
- Optimal relaxed solution is the optimal feasible solution of the relaxed problem.

Observation

Let \mathbf{x}^* be an integral optimal solution and \mathbf{x}' be a relaxed optimal solution. Then, $\mathbf{c}^T \mathbf{x}' \geq \mathbf{c}^T \mathbf{x}^*$.

Branch

Consider a mix integer linear programming problem
 $\max \{ \mathbf{x} \in \mathbb{R}^n; \mathbf{Ax} \leq \mathbf{b}, \mathbf{x}_i \in \mathbb{Z}, i \in I \}$ where I is a set of integral variables.

- Let \mathbf{x}^* be the optimal relaxed solution.
- If $\mathbf{x}_i^* \in \mathbb{Z}$ for all $i \in I$, then \mathbf{x}^* is an optimal solution.
- Otherwise, choose $j \in I$ such that $\mathbf{x}_j^* \notin \mathbb{Z}$ and
- recursively solve two subproblems
 - $\max \{ \mathbf{x} \in \mathbb{R}^n; \mathbf{Ax} \leq \mathbf{b}, \mathbf{x}_j \leq \lfloor \mathbf{x}_j^* \rfloor, \mathbf{x}_i \in \mathbb{Z}, i \in I \}$ and
 - $\max \{ \mathbf{x} \in \mathbb{R}^n; \mathbf{Ax} \leq \mathbf{b}, \mathbf{x}_j \geq \lceil \mathbf{x}_j^* \rceil, \mathbf{x}_i \in \mathbb{Z}, i \in I \}$.
- The optimal solution of the original problem is the better one of subproblems.

Bound

Let \mathbf{x}^* be an integral feasible solution and \mathbf{x}' be an optimal relaxed solution of a subproblem. If $\mathbf{c}^T \mathbf{x}' \geq \mathbf{c}^T \mathbf{x}^*$, then the subproblem does not contain better integral feasible solution than \mathbf{x}^* .

Observation

If the polyhedron $\{ \mathbf{x} \in \mathbb{R}^n; \mathbf{Ax} \leq \mathbf{b} \}$ is bounded, then the Branch and bound algorithm finds an optimal solution of the mix integer linear programming problem.

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Rational and integral polyhedrons

Theorem

A rational polytope P is integral if and only if for all integral vector \mathbf{c} the optimal value of $\max \{ \mathbf{c}^T \mathbf{x}; \mathbf{x} \in P \}$ is an integer.

Proof

- ⇒ Every vertex of P is integral, so optimal values are integrals.
- ⇐ Let \mathbf{v} be a vertex of P . We prove that \mathbf{v}_1 is an integer.
- Let \mathbf{c} be an integer vector such that \mathbf{v} is the only optimal solution.
 - Since we can scale the vector \mathbf{c} , we assume that $\mathbf{c}^T \mathbf{v} > \mathbf{c}^T \mathbf{u} + \mathbf{u}_1 - \mathbf{v}_1$ for all other vertices \mathbf{u} of P .
 - Let $\mathbf{d} = \mathbf{c} + \mathbf{e}_1$.
 - Observe that \mathbf{v} is an optimal solution of $\max \{ \mathbf{d}^T \mathbf{x}; \mathbf{x} \in P \}$.
 - Hence, $\mathbf{v}_1 = \mathbf{d}^T \mathbf{v} - \mathbf{c}^T \mathbf{v}$ is an integer.

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Gomory-Chvátal cutting plane proof

System of inequalities

Consider a system $P = \{ \mathbf{x}; \mathbf{Ax} \leq \mathbf{b} \}$ with n variables and m inequalities.

Definition: Gomory-Chvátal cutting plane

- Consider a non-negative linear combination of inequalities $\mathbf{y} \in \mathbb{R}^m$
- Let $\mathbf{c} = \mathbf{y}^T \mathbf{A}$ and $d = \mathbf{y}^T \mathbf{b}$
- Every point $\mathbf{x} \in P$ satisfies $\mathbf{c}^T \mathbf{x} \leq d$
- Furthermore, if \mathbf{c} is integral, every integral point \mathbf{x} satisfies $\mathbf{c}^T \mathbf{x} \leq \lfloor d \rfloor$
- The inequality $\mathbf{c}^T \mathbf{x} \leq \lfloor d \rfloor$ is called a Gomory-Chvátal cutting plane

Definition: Gomory-Chvátal cutting plane proof

A cutting plane proof of an inequality $\mathbf{w}^T \mathbf{x} \leq t$ is a sequence of inequalities $\mathbf{a}_{m+k}^T \mathbf{x} \leq b_{m+k}$ where $k = 1, \dots, M$ such that

- for each $k = 1, \dots, M$ the inequality $\mathbf{a}_{m+k}^T \mathbf{x} \leq b_{m+k}$ is a cutting plane derived from the system $\mathbf{a}_i^T \mathbf{x} \leq b_i$ for $i = 1, \dots, m + k - 1$ and
- $\mathbf{w}^T \mathbf{x} \leq t$ is the last inequality $\mathbf{a}_{m+M}^T \mathbf{x} \leq b_{m+M}$.

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Gomory-Chvátal cutting plane: Proof of the lemma

Lemma

Let F be a face of a rational polytope P . If $\mathbf{c}^T \mathbf{x} \leq \lfloor d \rfloor$ is a cutting plane for F , then there exists a cutting plane $\mathbf{c}^T \mathbf{x} \leq d'$ such that

$$F \cap \{ \mathbf{x}; \mathbf{c}^T \mathbf{x} \leq \lfloor d' \rfloor \} = F \cap \{ \mathbf{x}; \mathbf{c}^T \mathbf{x} \leq \lfloor d \rfloor \}.$$

Proof

- Let $P = \{ \mathbf{x}; \mathbf{A}'\mathbf{x} \leq \mathbf{b}', \mathbf{A}''\mathbf{x} \leq \mathbf{b}'' \}$ and $F = \{ \mathbf{x}; \mathbf{A}'\mathbf{x} \leq \mathbf{b}', \mathbf{A}''\mathbf{x} = \mathbf{b}'' \}$ where \mathbf{A}' and \mathbf{b}' are integral
- Assume $d = \max \{ \mathbf{c}^T \mathbf{x}; \mathbf{x} \in F \}$
- By Farkas' lemma, there exists vectors $\mathbf{y}' \geq \mathbf{0}$ and \mathbf{y}'' such that $\mathbf{y}'^T \mathbf{A}' + \mathbf{y}''^T \mathbf{A}'' = \mathbf{c}^T$ and $\mathbf{y}'^T \mathbf{b}' + \mathbf{y}''^T \mathbf{b}'' = d$
- $\mathbf{c}' = \mathbf{c} - \lfloor \mathbf{y}'' \rfloor^T \mathbf{A}'' = \mathbf{y}'^T \mathbf{A}' + (\mathbf{y}'' - \lfloor \mathbf{y}'' \rfloor)^T \mathbf{A}''$
 $d' = d - \lfloor \mathbf{y}'' \rfloor^T \mathbf{b}'' = \mathbf{y}'^T \mathbf{b}' + (\mathbf{y}'' - \lfloor \mathbf{y}'' \rfloor)^T \mathbf{b}''$
- Since \mathbf{y}' and $(\mathbf{y}'' - \lfloor \mathbf{y}'' \rfloor)^T$ are non-negative, $\mathbf{c}'^T \mathbf{x} \leq d'$ is a valid inequality for P
- Hence, $F \cap \{ \mathbf{x}; \mathbf{c}'^T \mathbf{x} \leq \lfloor d' \rfloor \} = F \cap \{ \mathbf{x}; \mathbf{c}'^T \mathbf{x} \leq \lfloor d' \rfloor, \lfloor \mathbf{y}'' \rfloor^T \mathbf{A}'' \mathbf{x} = \lfloor \mathbf{y}'' \rfloor^T \mathbf{b}'' \} = F \cap \{ \mathbf{x}; \mathbf{c}^T \mathbf{x} \leq \lfloor d \rfloor \}$.

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Definition: Rational polyhedron

A polyhedron is called rational if it is defined by a rational linear system, that is $\mathbf{A} \in \mathbb{Q}^{m \times n}$ and $\mathbf{b} \in \mathbb{Q}^m$.

Exercise

Every vertex of a rational polyhedron is rational.

Definition: Integral polyhedron

A rational polyhedron is called integral if every non-empty face contains an integral point.

Observation

Let P be a rational polyhedron which has a vertex. Then, P is integral if and only if every vertex of P is integral.

Theorem

A rational polytope P is integral if and only if for all integral vector \mathbf{c} the optimal value of $\max \{ \mathbf{c}^T \mathbf{x}; \mathbf{x} \in P \}$ is an integer.

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Gomory-Chvátal cutting plane: Example

Integer linear programming problem

$$\begin{array}{llll} \text{Maximize} & & \mathbf{x}_2 & \\ \text{subject to} & 2\mathbf{x}_1 + 3\mathbf{x}_2 & \leq & 27 \\ & 2\mathbf{x}_1 - 2\mathbf{x}_2 & \leq & 7 \\ & -2\mathbf{x}_1 - 6\mathbf{x}_2 & \leq & -11 \\ & -6\mathbf{x}_1 + 8\mathbf{x}_2 & \leq & 21 \\ & \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{Z} & & \end{array}$$

Relaxed problem

Optimal relaxed solution is $(\frac{9}{2}, 6)^T$.

Cutting plane 1

$$\begin{array}{ll} \text{The last inequality} & -3\mathbf{x}_1 + 4\mathbf{x}_2 \leq \frac{21}{2} \\ \text{Every feasible } \mathbf{x} \in \mathbb{Z}^2 \text{ satisfies} & -3\mathbf{x}_1 + 4\mathbf{x}_2 \leq 10 \end{array}$$

Cutting plane 2

$$\begin{array}{ll} \text{Cutting plane 1} & -6\mathbf{x}_1 + 8\mathbf{x}_2 \leq 20 \\ \text{The first inequality} & 6\mathbf{x}_1 + 9\mathbf{x}_2 \leq 81 \\ \text{Sum} & 17\mathbf{x}_2 \leq 101 \\ \text{Every feasible } \mathbf{x} \in \mathbb{Z}^2 \text{ satisfies} & \mathbf{x}_2 \leq 5 \end{array}$$

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Gomory-Chvátal cutting plane: Theorems

Theorem: Existence of a cutting plane proof for every valid inequality

Let $P = \{ \mathbf{x}; \mathbf{Ax} \leq \mathbf{b} \}$ be a rational polytope and let $\mathbf{w}^T \mathbf{x} \leq t$ be an inequality with \mathbf{w}^T integral satisfied by all integral vectors in P . Then there exists a cutting plane proof of $\mathbf{w}^T \mathbf{x} \leq t'$ from $\mathbf{Ax} \leq \mathbf{b}$ for some $t' \leq t$.

Theorem: Cutting plane proof for $\mathbf{0}^T \mathbf{x} \leq -1$ in polytopes without integral point

Let $P = \{ \mathbf{x}; \mathbf{Ax} \leq \mathbf{b} \}$ be a rational polytope that contains no integral point. Then there exists a cutting plane proof of $\mathbf{0}^T \mathbf{x} \leq -1$ from $\mathbf{Ax} \leq \mathbf{b}$.

Lemma

Let F be a face of a rational polytope P . If $\mathbf{c}^T \mathbf{x} \leq \lfloor d \rfloor$ is a cutting plane for F , then there exists a cutting plane $\mathbf{c}^T \mathbf{x} \leq d'$ such that

$$F \cap \{ \mathbf{x}; \mathbf{c}^T \mathbf{x} \leq \lfloor d' \rfloor \} = F \cap \{ \mathbf{x}; \mathbf{c}^T \mathbf{x} \leq \lfloor d \rfloor \}.$$

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Gomory-Chvátal cutting plane: Proof of the Theorem

Theorem: Cutting plane proof for $\mathbf{0}^T \mathbf{x} \leq -1$ in polytopes without integral point

Let $P = \{ \mathbf{x}; \mathbf{Ax} \leq \mathbf{b} \}$ be a rational polytope that contains no integral point. Then there exists a cutting plane proof of $\mathbf{0}^T \mathbf{x} \leq -1$ from $\mathbf{Ax} \leq \mathbf{b}$.

Proof

Induction by $\dim(P)$. Trivial for $\dim(P) = 0$. Assume $\dim(P) \geq 1$.

- Let $\mathbf{w}^T \mathbf{x} \leq l$ induces a proper face of P and $\bar{P} = \{ \mathbf{x} \in P; \mathbf{w}^T \mathbf{x} \leq \lfloor l \rfloor \}$
- We derive $\mathbf{0}^T \mathbf{x} \leq -1$ from $\mathbf{Ax} \leq \mathbf{b}$, $\mathbf{w}^T \mathbf{x} \leq \lfloor l \rfloor$ by the following two cases
 - If $\bar{P} = \emptyset$, we use Farkas' lemma
 - If $\bar{P} \neq \emptyset$, let $F = \{ \mathbf{x} \in P; \mathbf{w}^T \mathbf{x} = \lfloor l \rfloor \}$
 - Since $\dim(F) < \dim(P)$, there exists a cutting plane proof of $\mathbf{0}^T \mathbf{x} \leq -1$ from $\mathbf{Ax} \leq \mathbf{b}$, $\mathbf{w}^T \mathbf{x} = \lfloor l \rfloor$
 - By lemma, there exists a cutting plane proof of $\mathbf{c}^T \mathbf{x} \leq \lfloor d \rfloor$ such that $\bar{P} \cap \{ \mathbf{x}; \mathbf{c}^T \mathbf{x} \leq \lfloor d \rfloor, \mathbf{w}^T \mathbf{x} = \lfloor l \rfloor \} = \emptyset$
 - Applying these sequence of cuts to \bar{P} , we obtain $\mathbf{w}^T \mathbf{x} \leq \lfloor l \rfloor - 1$
 - Repeat these steps on $\bar{P} = \{ \mathbf{x} \in P; \mathbf{w}^T \mathbf{x} \leq \lfloor l \rfloor - 1 \}$
- The number of repetitions is finite since P is bounded

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Theorem: Existence of a cutting plane proof for every valid inequality

Let $P = \{x; Ax \leq b\}$ be a rational polytope and let $w^T x \leq t$ be an inequality with w^T integral satisfied by all integral vectors in P . Then there exists a cutting plane proof of $w^T x \leq t'$ from $Ax \leq b$ for some $t' \leq t$.

Proof

Let $I = \max \{w^T x; x \in P\}$ and $\bar{P} = \{x \in P; w^T x \leq \lfloor I \rfloor\}$

- If P contains no integer point, then there exists a cutting plane proof of $0^T x \leq -1$ and $w^T x \leq t'$ for some $t' \leq t$
- If P contains an integer point, then:
 - 1 If $\lfloor I \rfloor \leq t$, we are finished, so we suppose not
 - 2 $F = \{x \in P; w^T x = \lfloor I \rfloor\}$ is a face of P
 - 3 Since F has no integral point, we derive $0^T x \leq -1$ from $Ax \leq b, w^T x = \lfloor I \rfloor$
 - 4 By lemma, there exists a cutting plane proof of $c^T x \leq \lfloor d \rfloor$ from $Ax \leq b, w^T x \leq \lfloor I \rfloor$ such that $P \cap \{x; c^T x \leq \lfloor d \rfloor, w^T x = \lfloor I \rfloor\} = \emptyset$
 - 5 We apply this sequence of cuts to \bar{P} to obtain cutting plane $w^T x \leq \lfloor I \rfloor - 1$
 - 6 Now, we continue until we derive $w^T x \leq t'$ for some $t' \leq t$

Unimodular matrix

Definition

A full row rank matrix A is unimodular if A is integral and each basis of A has determinant ± 1 .

Theorem

Let $A \in \mathbb{R}^{m \times n}$ be an integral full row rank matrix. Then, the polyhedron $P = \{x; Ax = b, x \geq 0\}$ is integral for every integral vector b if and only if A is unimodular.

Proof

- \Leftarrow
 - Let b be an integral vector and let x' be a vertex of P
 - Columns of A corresponding to non-zero components of x' are linearly independent and we extend these columns into a basis A_B
 - Hence, $x'_B = A_B^{-1}b$ is integral and $x'_N = 0$
- \Rightarrow
 - We prove that $A_B^{-1}v$ is integral for every base B and integral vector v
 - Let y be integral vector such that $y + A_B^{-1}v \geq 0$
 - Let $b = A_B(y + A_B^{-1}v) = A_B y + v$ which is integral
 - Let $z_B = y + B^{-1}v$ and $z_N = 0$
 - From $Az = A_B(y + B^{-1}v) = b$ and $z \geq 0$, it follows that $z \in P$ and z is a vertex of P
 - Hence, $A_B^{-1}v = z_B - y$ is integral

Totally unimodular matrix

Theorem: Hoffman-Kruskal

Let $A \in \mathbb{Z}^{m \times n}$ and $P = \{x; Ax \leq b, x \geq 0\}$. The polyhedron P is integral for every integral b if and only if A is totally unimodular.

Proof

Adding slack variables, we observe that the following statements are equivalent.

- $\{x; Ax \leq b, x \geq 0\}$ is integral for every integral b
- $\{x; (A|I)z = b, z \geq 0\}$ is integral for every integral b
- $(A|I)$ is unimodular
- A is totally unimodular

Theorem

Let A be an totally unimodular matrix and let b be an integral vector. Then, The polyhedron defined by $Ax \leq b$ is integral.

Proof

- Let $F = \{x; A'x \leq b', A''x = b''\}$ be a minimal face where A'' has full row rank
- Let B be a basis of A''
- Then, $x_B = A_B'^{-1}b'$ and $x_N = 0$ is an integral point in F

Network flow

Definition: Network flow

Let $G = (V, E)$ be an oriented graph with non-negative capacities of edges $c \in \mathbb{R}^E$. A network flow in G is a vector $f \in \mathbb{R}^E$ such that

Conservation: $\sum_{uv \in E} f_{uv} = \sum_{vu \in E} f_{vu}$ for every vertex $v \in V$

Capacity: $0 \leq f \leq c$

The network flow problem is the optimization problem of finding a flow f in G that maximize f_{ts} on a given edge $ts \in E$.

Theorem

The polytope of network flow is integral for every integral c .

Proof

- Let A be the incidence matrix of G
- A is totally unimodular
- $(A| -A)$ and $(A| -A|I)$ are totally unimodular
- $\left\{ f; \begin{pmatrix} A \\ -A \\ I \end{pmatrix} f \leq \begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix}, f \geq 0 \right\}$ is an integral polytope

Questions

- How to recognise whether a polytope $P = \{x; Ax \leq b\}$ is integral?
- When P is integral for every integral vector b ?

Proposition

Let $A \in \mathbb{R}^{m \times m}$ be an integral and regular matrix. Then, $A^{-1}b$ is integral for every integral vector $b \in \mathbb{R}^m$ if and only if $\det(A) \in \{1, -1\}$.

Proof

- \Leftarrow By Cramer's rule, A^{-1} is integral, so $A^{-1}b$ is integral for every integral b
- \Rightarrow
 - $A_{*,i}^{-1} = A^{-1}e_i$ is integral for every $i = 1, \dots, m$
 - Since A and A^{-1} are integral, also $\det(A)$ and $\det(A^{-1})$ are both integers
 - From $1 = \det(A) \cdot \det(A^{-1})$ it follows that $\det(A) = \det(A^{-1}) \in \{1, -1\}$

Totally unimodular matrix

Definition

A matrix is totally unimodular if all of its square submatrices have determinant 0, 1 or -1 .

Exercise

Prove that every element of a totally unimodular matrix is 0, 1 or -1 . Find a matrix $A \in \{0, 1, -1\}^{m \times n}$ which is not totally unimodular.

Exercise

Prove that A is totally unimodular if and only if $(A|I)$ is unimodular.

Totally unimodular matrix: Application

Observation

Let A be a matrix of 0, 1 and -1 where every column has at most one $+1$ and at most one -1 . Then, A is totally unimodular.

Proof

By the induction on k prove that every $k \times k$ submatrix N has determinant 0, $+1$ or -1

$k = 1$ Trivial

- $k > 1$
 - If N has a column with at most one non-zero element, then we expand this column and use induction
 - If N has exactly one $+1$ and -1 in every column, then the sum of all rows is 0, so N is singular

Corollary

The incidence matrix of an oriented graph is totally unimodular.

Observation: Other totally unimodular (TU) matrices

A is TU iff A^T is TU iff $(A|I)$ is TU iff $(A|A)$ is TU iff $(A| -A)$ is TU

Duality of the network flow problem

Primal: Network flow

Maximize f_{ts} subject to $Af = 0, f \leq c$ and $f \geq 0$.

Primal dual

Minimize cz subject to $A^T y + z \geq e_{ts}$, that is $-y_u + y_v + z_{uv} \geq 0$ for $uv \neq ts$ and $-y_t + y_s \geq 1$ assuming $f(ts)$ is unbounded.

Observation

Dual problem has an integral optimal solution.

Theorem

The dual problem is the minimal cut problem where $Z = \{uv \in E; z_{uv} = 1\}$ are cut edges and $U = \{u \in V; y_u > y_t\}$ is partition of vertices.

- 1 Linear programming
- 2 Linear, affine and convex sets
- 3 Simplex method
- 4 Duality of linear programming
- 5 Integer linear programming
- 6 Matching
- 7 Ellipsoid method
- 8 Vertex Cover
- 9 Matroid

Tutte-Berge Formula

Definition

Let $\text{def}(G)$ be the number of exposed edges by a maximal-size matching in $G = (V, E)$.

Definition

Let $\text{oc}(G)$ be the number of odd components of a graph G .

Observation

For every $A \subseteq V$ it holds that $\text{def}(G) \geq \text{oc}(G \setminus A) - |A|$.

Theorem: Tutte-Berge Formula

$$\text{def}(G) = \min \{ \text{oc}(G \setminus A) - |A| : A \subseteq V \}$$

Proof

- ≥ Follows from the previous observation.
- ≤ An algorithm presented later.

Tutte's matching theorem

A graph G has a perfect matching if and only if $\text{oc}(G \setminus A) \leq |A|$ for every $A \subseteq V$.

Operations

Use $uv \in E$ to extend T

Input: A matching M of a graph G , an M -alternating tree T , edge $uv \in E$ such that $u \in B$ and $v \notin A \cup B$ and v is M -covered

Action: Let $vz \in M$ and extend T by edges uv and vz

Use $uv \in E$ to augment M

Input: A matching M of a graph G , an M -alternating tree T with root r , edge $uv \in E$ such that $u \in B$ and $v \notin A \cup B$ and v is M -exposed

Action: Let P be the path obtained by attaching uv to the path from r to v in T . Replace M by $M \Delta E(P)$.

Perfect matchings in bipartite graphs

Minimal-weight perfect matching

Let G be a graph with weights $c \geq 0$ on edges. The minimal-weight perfect matching problem is minimizing $\mathbf{c}\mathbf{x}$ subject to $A\mathbf{x} = \mathbf{1}$ and $\mathbf{x} \in \{0, 1\}^E$ where A is the incidence matrix.

Observation

The incidence matrix A of a bipartite graph G is totally unimodular.

Proof

By the induction on k prove that every $k \times k$ submatrix N has determinant 0, +1 or -1

$k = 1$ Trivial

- $k > 1$
 - If N has a column or a row with at most one non-zero element, then we expand this column and use induction
 - Otherwise, the subgraph of edges corresponding to rows of N contains a cycle and rows corresponding to edges of a cycle are linearly dependent.

Theorem

If A is an incidence matrix of a bipartite graph, then $\{\mathbf{x} : A\mathbf{x} = \mathbf{1}, \mathbf{x} \geq 0\}$ is integral.

Augmenting paths

Definitions

Let $M \subseteq E$ a matching of a graph $G = (V, E)$.

- A vertex $v \in V$ is M -covered if some edge of M is incident with v .
- A vertex $v \in V$ is M -exposed if v is not M -covered.
- A path P is M -alternating if its edges are alternately in and not in M .
- An M -alternating path is M -augmenting if both end-vertices are M -exposed.

Augmenting path theorem of matchings

A matching M in a graph $G = (V, E)$ is maximum if and only if there is no M -augmenting path.

Proof

⇒ Every M -augmenting path increases the size of M

- ⇐ Let N be a matching such that $|N| > |M|$ and we find an M -augmenting path
 - 1 The graph $(V, N \cup M)$ contains a component K which has more N edges than M edges
 - 2 K has at least two vertices u and v which are N -covered and M -exposed
 - 3 Vertices u and v are joined by a path P in K
 - 4 Observe that P is M -augmenting

Alternating tree

Construction of an M -alternating tree T on vertices $A \cup B$

Init: $A = \emptyset$ and $B = \{r\}$ where r is an M -exposed root

Step: Let $uv \in E$ such that $u \in B$, $v \notin A \cup B$ and $vz \in M$ for some $z \in V$. Add v to A and z to B

Properties

- r is the only M -exposed vertex of T
- For every v of T , the path in T from v to r is M -alternating

Definition

M -alternation path T is frustrated if every edge of G having one edge in B has the other end in A

Observation

If G has a matching M and a frustrated M -alternating tree, then G has no perfect matching.

Proof

B are single vertex components of $G \setminus A$, so $\text{oc}(G \setminus A) \geq |B| > |A|$

Perfect matchings algorithm in a non-weighted bipartite graph

Algorithm

```

1 Init:  $M = \emptyset$  and  $T = (\{r\}, \emptyset)$  where  $r$  is an arbitrary vertex
2 while there exists  $uv \in E$  with  $u \in B(T)$  and  $v \notin V(T)$  do
3   if  $v$  is  $M$ -exposed then
4     Use  $uv$  to augment  $M$ 
5     if there is no  $M$ -exposed node in  $G$  then
6       return  $M$ 
7   else
8     Replace  $T$  by  $(\{r\}, \emptyset)$  where  $r$  is an  $M$ -exposed vertex
9   else
10    Use  $uv$  to extend  $T$ 
11 return  $G$  has no perfect matching since  $T$  is a frustrated  $M$ -alternating path

```

Theorem

The algorithm decides whether a given bipartite graph G has a perfect matching and find one if exists. The algorithm calls $O(n)$ augmenting operations and $O(n^2)$ extending operations.

Duality and complementary slackness of perfect matchings

Primal: relaxed perfect matching

Minimize $\mathbf{c}^T \mathbf{x}$ subject to $A\mathbf{x} = \mathbf{1}$ and $\mathbf{x} \geq 0$.

Dual

Maximize $\mathbf{1}^T \mathbf{y}$ subject to $A^T \mathbf{y} \leq \mathbf{c}$ and $\mathbf{y} \in \mathbb{R}^E$, that is $\mathbf{y}_u + \mathbf{y}_v \leq \mathbf{c}_{uv}$.

Idea of primal-dual algorithms

If we find a primal and a dual feasible solutions satisfying the complementary slackness, then solutions are optimal (relaxed) solutions.

Definition

- An edge $uv \in E$ is called *tight* if $\mathbf{y}_u + \mathbf{y}_v = \mathbf{c}_{uv}$.
- Let E_y be the set of a tight edges of the dual solution \mathbf{y} .
- Let $M_x = \{uv \in E : \mathbf{x}_{uv} = 1\}$ be the set of matching edge of the primal solution \mathbf{x} .

Complementary slackness

$\mathbf{x}_{uv} = 0$ or $\mathbf{y}_u + \mathbf{y}_v = \mathbf{c}_{uv}$ for every edge $uv \in E$, that is $M_x \subseteq E_y$

Weighted perfect matchings in a bipartite graph: Overview

Complementary slackness

$x_{uv} = 0$ or $y_u + y_v = c_{uv}$ for every edge $uv \in E$, that is $M_x \subseteq E_y$

Invariants

- Dual solution is feasible, that is $y_u + y_v \leq c_{uv}$
- Every matching edge is tight
- $x \in \{0, 1\}^E$ and $M_x = \{uv \in E; x_{uv} = 1\}$ form a matching

Initial solution satisfying invariants

$x = 0$ and $y = 0$

Lemma: optimality

If M_x is a perfect matching, then M_x is a perfect matching with the minimal weight.

Idea of the algorithm

- If there exists an M_x -augmenting path P in (V, E_y) , then $M_x \triangle P$ is a new matching.
- Otherwise, update the dual solution y to enlarge E_y .

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Shrinking odd circuits

Definition

Let C be an odd circuit in G . The graph $G \times C$ has vertices $(V(G) \setminus V(C)) \cup \{c'\}$ where c' is a new vertex and edges

- $E(G)$ with both end-vertices in $V(G) \setminus V(C)$ and
- and uc' for every edge uv with $u \notin V(C)$ and $v \in V(C)$.

Edges $E(C)$ are removed.

Proposition

Let C be an odd circuit of G and M' be a matching $G \times C$. Then, there exists a matching M of G such that $M \subseteq M' \cup E(C)$ and the number of M' -exposed nodes of G is the same as the number of M' -exposed nodes in $G \times C$.

Corollary

$\text{def}(G) \leq \text{def}(G \times C)$

Exercise

Find a graph G with odd circuit C such that $\text{def}(G) < \text{def}(G \times C)$.

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Perfect matchings algorithm in a non-weighted graph

Algorithm

```

1 Init:  $M' = M = \emptyset$ ,  $G' = G$  and  $T = (\{r\}, \emptyset)$  where  $r$  is an arbitrary vertex
2 while there exists  $uv \in E'$  with  $u \in B$  and  $v \notin T$  do
3   if  $v \notin T$  is  $M'$ -exposed then
4     Use  $uv$  to augment  $M'$ 
5     Extend  $M'$  to a matching  $G$ 
6     Replace  $M'$  by  $M$  and  $G'$  by  $G$ 
7     If there is no  $M'$ -exposed node in  $G'$  then
8       return Perfect matching  $M$ 
9   else
10    Replace  $T$  by  $(\{r\}, \emptyset)$  where  $r$  is an  $M'$ -exposed vertex
11  else if  $v \notin T$  is  $M'$ -covered then
12    Use  $uv$  to extend  $T$ 
13  else if  $v \in B$  then
14    Use  $uv$  to shrink and update  $M'$  and  $T$ 
15 return  $G$  has no perfect matching since  $T$  is a frustrated  $M$ -alternating path
    
```

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Minimum-Weight perfect matchings in general graphs: Duality

Primal

Minimize $\mathbf{c}x$
 subject to $\delta^u x = 1$ for all $u \in V$
 $\delta^D x \geq 1$ for all $D \in C$
 $x \geq 0$

Dual

Maximize $\sum_{v \in V} y_v + \sum_{D \in C} z_D$
 subject to $y_u + y_v + \sum_{uv \in D \in C} z_D \leq c_{uv}$ for all $uv \in E$
 $z \geq 0$

Notation: Reduced cost

$\bar{c}_{uv} := c_{uv} - y_u - y_v - \sum_{uv \in D \in C} z_D$
 An edge e is tight if $\bar{c}_e = 0$

Complementary slackness

- $x_e > 0$ implies e is tight for all $e \in E$
- $z_D > 0$ implies $\delta^D x = 1$ for all $D \in C$

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Minimal weight perfect matchings algorithm in a bipartite graph

Algorithm

```

1 Init:  $y = 0$  and  $M = \emptyset$  and  $T = (\{r\}, \emptyset)$  where  $r$  is an arbitrary vertex
2 Loop
3   Find a perfect matching  $M$  in  $(V, E_y)$  or frustrated  $M$ -alternating tree
4   if  $M$  is a perfect matching of  $G$  then
5     return Perfect matching  $M$ 
6    $\epsilon = \min \{c_{uv} - y_u - y_v; u, v \in E, u \in B(T), v \notin T\}$ 
7   if  $\epsilon = \infty$  then
8     return Dual problem is unbounded, so there is no perfect matching
9    $y_u := y_u + \epsilon$  for all  $u \in B$ 
10   $y_v := y_v - \epsilon$  for all  $v \in A$ 
    
```

Theorem

The algorithm decides whether a given bipartite graph G has a perfect matching and a minimal-weight perfect matching if exists. The algorithm calls $O(n)$ augmenting operations and $O(n^2)$ extending operations and $O(n^2)$ dual changes.

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Perfect matching in general graphs

Use uv to shrink and update M' and T

Input: A matching M' of a graph G' , an M' -alternating tree T , edge $uv \in E'$ such that $u, v \in B$

Action: Let C be the circuit formed by uv together with the path in T from u to v . Replace G' by $G' \times C$, M' by $M' \setminus E(C)$ and T by the tree having edge-set $E(T) \setminus E(C)$.

Observation

Let G' be a graph obtained from G by a sequence of odd-circuit shrinkings. Let M' be matching of G' and let T be an M' -alternating tree of G' such that all vertices of A are original vertices of G . If T is frustrated, then G has no perfect matching.

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Minimum-Weight perfect matchings in general graphs

Observation

Let M be a perfect matching of G and D be an odd set of vertices of G . Then there exists at least one edge $uv \in M$ between D and $V \setminus D$.

Linear programming for Minimum-Weight perfect matchings in general graphs

Minimize $\mathbf{c}x$
 subject to $\delta^u x = 1$ for all $u \in V$
 $\delta^D x \geq 1$ for all $D \in C$
 $x \geq 0$

Where $\delta^D \in \{0, 1\}^E$ is a vector such that $\delta_{uv}^D = 1$ if $|uv \cap D| = 1$ and $\delta^w = \delta^{\{w\}}$ and C is the set of all odd-size subsets of V .

Exercise

Find a cutting plane proof of all odd-subset conditions.

Theorem

Let G be a graph and $\mathbf{c} \in \mathbb{R}^E$. Then G has a perfect matching if and only if the LP problem is feasible. Moreover, if G has a perfect matching, the minimum weight of a perfect matching is equal to the optimal value of the LP problem.

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Minimum-Weight perfect matchings in general graphs: Change of y

Updates weights and dual solution when shrinking a circuit C

Replace c'_{uv} by $c'_{uv} - y'_u - y'_v$ for $u \in C$ and $v \notin C$ and set $y'_{c'} = 0$ for the new vertex c' . Note that the reduced cost is unchanged.

Expand c' into circuit C

- Set $z'_C = y'_{c'}$
- Replace c'_{uv} by $c'_{uv} + y'_u + y'_v$ for $u \in C$ and $v \notin C$
- Update M' and T

Change of y and z on a frustrated tree

Input: A graph G' with weights \mathbf{c}' , a feasible dual solution \mathbf{y}' , a matching M' of tight edges of G' and an M' -alternating tree T of tight edges of G' .

Action:

- $\epsilon_1 = \min \{\bar{c}'_e; e \text{ joins a vertex in } B \text{ and a vertex not in } T\}$
- $\epsilon_2 = \min \{\bar{c}'_e/2; e \text{ joins two vertices of } B\}$
- $\epsilon_3 = \min \{y'_v; v \in A \text{ and } v \text{ is a pseudonode of } G\}$
- $\epsilon = \min \{\epsilon_1, \epsilon_2, \epsilon_3\}$
- Replace y'_v by $y'_v + \epsilon$ for all $v \in B$
- Replace y'_v by $y'_v - \epsilon$ for all $v \in A$

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Algorithm

```

1 Init:  $M' = M = \emptyset$ ,  $G' = G$  and  $T = (\{r\}, \emptyset)$  where  $r$  is an arbitrary vertex
2 Loop
3   if there exists  $uv \in E_y$ ,  $u \in B$ ,  $v \notin E(T)$ ,  $v$  is  $M'$ -exposed then
4     Use  $uv$  to augment  $M'$ 
5     if there is no  $M'$ -exposed node then
6       return extended  $M'$  to a perfect matching  $G$ 
7     else
8       Replace  $T$  by  $(\{r\}, \emptyset)$  where  $r$  is an  $M'$ -exposed vertex
9   else if there exists  $uv \in E_y$ ,  $u \in B$ ,  $v \notin E(T)$ ,  $v$  is  $M'$ -covered then
10     Use  $uv$  to extend  $T'$ 
11   else if there exists  $uv \in E_y$ ,  $u, v \in B$  then
12     Use  $uv$  to shrink and update  $M'$ ,  $T'$ ,  $c'$ 
13   else if there exists a pseudonode  $v \in A$  with  $y_v = 0$  then
14     Expand  $v$  and update  $M'$ ,  $T$ , and  $c'$ 
15   else
16     Change  $y$ 
17     if  $\epsilon = \infty$  then
18       return  $G$  has no perfect matching

```

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- 1 Linear programming
- 2 Linear, affine and convex sets
- 3 Simplex method
- 4 Duality of linear programming
- 5 Integer linear programming
- 6 Matching
- 7 **Ellipsoid method**
- 8 Vertex Cover
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Ellipsoid method

Idea

Consider an ellipsoid E containing Z . In every step, reduce the volume of E using an hyperplane provided by the oracle.

Algorithm

```

1 Init:  $s = \mathbf{0}$ ,  $E = B(s, R)$ 
2 Loop
3   if volume of  $E$  is smaller than volume of  $B(0, \epsilon)$  then
4     return  $Z$  is empty
5   Call the oracle
6   if  $s \in Z$  then
7     return  $s$  is a point of  $Z$ 
8   Update  $s$  and  $Z$  using the separation hyperplane found by oracle

```

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Ellipsoid method: update of the ellipsoid

Separation hyperplane

Consider a hyperplane $\mathbf{a}^T \mathbf{x} = b$ such that $\mathbf{a}^T \mathbf{s} \geq b$ and $Z \subseteq \{\mathbf{x}; \mathbf{a}^T \mathbf{x} \leq b\}$. For simplicity, assume that the hyperplane contains \mathbf{s} , that is $\mathbf{a}^T \mathbf{s} = b$.

Update formulas (without proof)

$$\mathbf{s}' = \mathbf{s} - \frac{1}{n+1} \frac{Q\mathbf{a}}{\sqrt{\mathbf{a}^T Q \mathbf{a}}}$$

$$Q' = \frac{n^2}{n^2 - 1} \left(Q - \frac{2}{n+1} \frac{Q\mathbf{a}\mathbf{a}^T Q}{\mathbf{a}^T Q \mathbf{a}} \right)$$

Reduce of the volume (without proof)

$$\frac{\text{volume}(E')}{\text{volume}(E)} \leq e^{-\frac{1}{2n+2}}$$

Corollary

The number of steps of the Ellipsoid method is at most $\lceil n(2n+2) \ln \frac{R}{\epsilon} \rceil$.

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Reduction to perfect matching problem

Let G be a graph with non-negative weights \mathbf{c} .

- Let G_1 and G_2 be two copies of G
- Let P be a perfect matching between G_1 and G_2 joining copied vertices
- Let G^* be a graph of vertices $V(G_1) \cup V(G_2)$ and edges $E(G_1) \cup E(G_2) \cup P$
- For $e \in E(G_1) \cup E(G_2)$ let $\mathbf{c}^*(e)$ be the weight of the original edge e on G
- For $e \in P$ let $\mathbf{c}^*(e) = 0$

Theorem

The maximal weight of a perfect matching in G^* equals twice the maximal weight of a matching in G .

Note

For maximal-size matching, use weights $\mathbf{c} = \mathbf{1}$.

Tutte's matching theorem

A graph G has a perfect matching if and only if $oc(G \setminus A) \leq |A|$ for every $A \subseteq V$.

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Ellipsoid method: Preliminaries

Problem

Determine whether a given fully-dimensional convex compact set $Z \subseteq \mathbb{R}^n$ (e.g. a polytope) is non-empty and find a point in Z if exists.

Separation oracle

Separation oracle determines whether a point s belongs into Z . If $s \notin Z$, the oracle finds a hyperplane that separates s and Z .

Inputs

- Radius $R > 0$ of a ball $B(\mathbf{0}, R)$ containing Z
- Radius $\epsilon > 0$ such that Z contains $B(s, \epsilon)$ for some point s if Z is non-empty
- Separation oracle

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Ellipsoid

Definition: Ball

The ball in the centre $\mathbf{s} \in \mathbb{R}^n$ and radius $R \geq 0$ is $B(\mathbf{s}, R) = \{\mathbf{x} \in \mathbb{R}^n; \|\mathbf{x} - \mathbf{s}\| \leq R\}$.

Definition

Ellipsoid E is an affine transformation of the unit ball $B(\mathbf{0}, 1)$. That is, $E = \{M\mathbf{x} + \mathbf{s}; \mathbf{x} \in B(\mathbf{0}, 1)\}$ where M is a regular matrix and \mathbf{s} is the centre of E .

Notation

$$\begin{aligned} E &= \{\mathbf{y} \in \mathbb{R}^n; M^{-1}(\mathbf{y} - \mathbf{s}) \in B(\mathbf{0}, 1)\} \\ &= \{\mathbf{y} \in \mathbb{R}^n; (\mathbf{y} - \mathbf{s})^T (M^{-1})^T M^{-1} (\mathbf{y} - \mathbf{s}) \leq 1\} \\ &= \{\mathbf{y} \in \mathbb{R}^n; (\mathbf{y} - \mathbf{s})^T Q^{-1} (\mathbf{y} - \mathbf{s}) \leq 1\} \end{aligned}$$

where $Q = MM^T$ is a positive definite matrix

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Ellipsoid method: Estimation of radii for rational polytopes

Largest coefficient of A and \mathbf{b}

Let L be the maximal absolute value of all coefficients of A and \mathbf{b} .

Estimation of R

We find R' such that $\|\mathbf{x}\|_\infty \leq R'$ for all \mathbf{x} satisfying $A\mathbf{x} \leq \mathbf{b}$:

- Consider a vertex of the polytope satisfying a subsystem $A'\mathbf{x} = \mathbf{b}'$
- Cramer's rule: $x_i = \frac{\det A'_i}{\det A'}$
- $|\det(A')| \leq n! L^n$ using the definition of determinant
- $|\det(A')| \geq 1$ since A' is integral and regular

From the choice $R' = n! L^n$, it follows that $\log(R) = O(n^2 \log(n) \log(L))$

Estimation of ϵ (without proof)

A non-empty rational fully-dimensional polytope contains a ball with radius ϵ where $\log \frac{1}{\epsilon} = O(\text{poly}(n, m, \log L))$.

Complexity of Ellipsoid method

Time complexity of Ellipsoid method is polynomial in the length of binary encoding of A and \mathbf{b} .

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Ellipsoid method is not strongly polynomial (without proof)

For every M there exists a linear program with 2 variables and 2 constraints such that the ellipsoid method executes at least M mathematical operations.

Open problem

Decide whether there exist an algorithm for linear programming which is polynomial in the number of variables and constraints.

Minimum vertex cover problem

Definition

A vertex cover in a graph $G = (V, E)$ is a set of vertices S such that every edge of E has at least one end vertex in S . Finding a minimal-size vertex cover is the minimum vertex cover problem.

Integer linear programming formulation

$$\begin{array}{ll} \text{Minimize} & \sum_{v \in V} x_v \\ \text{subject to} & x_u + x_v \geq 1 \quad \text{for all } uv \in E \\ & x_v \in \{0, 1\} \quad \text{for all } v \in V \end{array}$$

Relaxed problem

$$\begin{array}{ll} \text{Minimize} & \sum_{v \in V} x_v \\ \text{subject to} & x_u + x_v \geq 1 \quad \text{for all } uv \in E \\ & 0 \leq x_v \leq 1 \quad \text{for all } v \in V \end{array}$$

Maximum independent set problem

Definition

An independent set in a graph $G = (V, E)$ is a set of vertices S such that every edge of E has at **most** one end vertex in S . Finding a maximal-size independent is the maximal independent problem.

Integer linear programming formulation

$$\begin{array}{ll} \text{Minimize} & \sum_{v \in V} x_v \\ \text{subject to} & x_u + x_v \leq 1 \quad \text{for all } uv \in E \\ & x_v \in \{0, 1\} \quad \text{for all } v \in V \end{array}$$

Relaxed problem

$$\begin{array}{ll} \text{Minimize} & \sum_{v \in V} x_v \\ \text{subject to} & x_u + x_v \leq 1 \quad \text{for all } uv \in E \\ & 0 \leq x_v \leq 1 \quad \text{for all } v \in V \end{array}$$

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- 1 Linear programming
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Approximation algorithm for vertex cover problem

Algorithm

- Let x^* the optimal relaxed solution
- Let $S_{LP} = \{v \in V; x_v^* \geq \frac{1}{2}\}$

Observation

S_{LP} is a vertex cover.

Observation

Let S_{OPT} be the minimal vertex cover. Then $\frac{|S_{LP}|}{|S_{OPT}|} \leq 2$.

Proof

- Since x^* is the optimal relaxed solution, $\sum_{v \in V} x_v^* \leq |S_{OPT}|$
- From the rounding rule, it follows that $|S_{LP}| \leq 2 \sum_{v \in V} x_v^*$
- Hence, $|S_{LP}| \leq 2 \sum_{v \in V} x_v^* \leq 2|S_{OPT}|$

Maximum independent set problem

Relaxed solution

The relaxed solution $x_v = \frac{1}{2}$ for all $v \in V$ is feasible, so the optimal relaxed solution is at least $\frac{n}{2}$.

Optimal integer solution

The maximal independent set of a complete graph K_n is a single vertex.

Conclusion

In general, an optimal integer solution can be far from an optimal relaxed solution and cannot be obtained by a simple rounding.

Inapproximability of the minimum independent set problem

Unless $P = NP$, for every C there is no polynomial-time approximation algorithm for the maximum independent set with the approximation error at most C .

Greedy algorithm for spanning tree problem

Definition

A subtree (V, J) of a connected graph (V, E) is called a spanning tree. Maximum-weight spanning tree is a problem to find the spanning of maximum weight.

Greedy algorithm for finding a non-weighted spanning tree

```

1 Init:  $J = \emptyset$ 
2 while there exists an edge  $e \notin J$  such that  $J \cup \{e\}$  is a forest do
3   Choose an arbitrary such  $e$ 
4   Replace  $J$  by  $J \cup \{e\}$ 

```

Greedy algorithm for finding a maximal-weight spanning tree

```

1 Init:  $J = \emptyset$ 
2 while there exists an edge  $e \notin J$  such that  $J \cup \{e\}$  is a forest do
3   Choose such  $e$  with maximum weight
4   Replace  $J$  by  $J \cup \{e\}$ 

```

General greedy algorithm

Family of subsets

Consider a finite set S with weights $c: S \rightarrow \mathbb{R}$ and a family of subsets $\mathcal{I} \subseteq 2^S$ called independent. Our problem is to find $A \in \mathcal{I}$

- with maximum cardinality or
- with maximum weight.

When the following algorithm finds the maximal subset?

```
1 Init:  $J = \emptyset$ 
2 while there exists an element  $e \in S \setminus J$  such that  $J \cup \{e\} \in \mathcal{I}$  do
3   Choose such  $e$  (with maximum weight)
4   Replace  $J$  by  $J \cup \{e\}$ 
```

Examples

- ✓ Spanning tree
- × Matching
- × Independent set of vertices

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Matroid and Greedy algorithm

Theorem

Let (S, \mathcal{I}) satisfies (M0) and (M1). Then the Greedy algorithm finds an optimal independent set of every $c \in \mathbb{R}^S$ if and only if (S, \mathcal{I}) is a matroid.

Proof

- ⇒
- For contradiction, consider $J \subseteq A \subseteq S$ such that $J \in \mathcal{I}$ is inclusion-maximal subset of A which is not cardinality-maximal
 - Let c be a characteristic vector of A
 - The Greedy algorithm may find J although it is not the maximal-weight independent set
- ⇐
- Let $J = \{e_1, \dots, e_m\}$ be found by the Greedy algorithm
 - Let $J' = \{q_1, \dots, q_l\}$ be an optimal solution
 - Let k be the least index with $c(q_k) > c(e_k)$
 - Let $A = \{e_1, \dots, e_{k-1}, q_1, \dots, q_k\}$
 - $\{e_1, \dots, e_{k-1}\} \subseteq J$ and $\{q_1, \dots, q_k\} \subseteq J'$ are independent by (M1)
 - $\{e_1, \dots, e_{k-1}, q_l\}$ is dependent for every $q_l \in \{q_1, \dots, q_k\} \setminus \{e_1, \dots, e_{k-1}\}$ since the Greedy algorithm does not choose q_l in the k -th step
 - Sets $A, \{e_1, \dots, e_{k-1}\}$ and $\{q_1, \dots, q_k\}$ contradict (M2)

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Equivalent definitions of a matroid

Theorem

A set system (S, \mathcal{I}) is a matroid if and only if

- (I0) $\emptyset \in \mathcal{I}$
- (I1) If $J' \subseteq J \in \mathcal{I}$, then $J' \in \mathcal{I}$
- (I2) For every $A, B \in \mathcal{I}$ with $|A| > |B|$ there exists $e \in A \setminus B$ such that $B \cup \{e\} \in \mathcal{I}$

Definition

A circuit of a set system (M, \mathcal{I}) is a minimal dependent set.

Observation

Let (S, \mathcal{I}) be a matroid, let $J \in \mathcal{I}$ and $e \in S$. Then $J \cup \{e\}$ contains at most one circuit.

Theorem

A set C of subsets of S is the set of circuits of a matroid if and only if

- (C0) $\emptyset \notin C$
- (C1) If $C_1, C_2 \in C$ and $C_1 \subseteq C_2$, then $C_1 = C_2$
- (C2) If $C_1, C_2 \in C$, $C_1 \neq C_2$ and $e \in C_1 \cap C_2$, then there exists $C' \in C$, $C \subseteq (C_1 \cup C_2) \setminus \{e\}$

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Matroid

Definition

A pair (S, \mathcal{I}) where S is a finite set and $\mathcal{I} \subseteq 2^S$ is called a matroid if

- (M0) $\emptyset \in \mathcal{I}$
 - (M1) If $J' \subseteq J \in \mathcal{I}$, then $J' \in \mathcal{I}$
 - (M2) For every $A \subseteq S$, every maximal independent subset of A has the same cardinality
- The cardinality of maximal independent subset of A is called rank $r(A)$.

Examples

- Forest matroid: S are edges of a graph and every forest is independent
- Linear matroid: S are vectors of a linear space and \mathcal{I} contains linearly independent vectors
- Uniform matroid: $\mathcal{I} = \{J \subseteq S; |J| \leq k\}$ for some k

Theorem

Let (S, \mathcal{I}) satisfies (M0) and (M1). Then the Greedy algorithm finds an optimal independent set of every $c \in \mathbb{R}^S$ if and only if (S, \mathcal{I}) is a matroid.

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Complexity of the Greedy algorithm

Inefficient

Enumerating whole \mathcal{I} is inefficient, e.g. providing all forests in the input.

Oracula

The input contains S and c and an oracula which decides whether a given $A \subseteq S$ is independent.

Complexity

Complexity is determined in the size of S and the number of calls of oracula.

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