Deadline for homeworks: 3.3.2016 at 9:00.

Problems 1-8 formulate using (Integer) Linear Programming both in the canonical and the equation forms.

Problem 1. A bakery produces four things: bread, bagels, baguettes and donuts. To bake a single bread, they need 500g of flour, 10 eggs and 50 grams of salt. To bake a bagel, they need 150 grams of flour, 2 eggs and 10g of salt. For a baguette, they need 230g of flour, 7 eggs and 15g of salt. For a donut, they need 100g of flour and 1 egg. The bakery has a daily supply of 5 kg of flour, 125 eggs, and 500g of salt.

The bakery charges 20 CZK for one bread, 2 CZK for a bagel, 10 CZK for a baguette and 7 CZK for a donut. The bakery tries to maximize its profit. Formulate a linear program that suggests the right amount of bread, bagels, baguettes and donuts that the bakery should produce. (You don't have to solve it!)

Problem 2 (Independent set)**.** A set of vertices S of a graph (V, E) is called *independent* if no two vertices of S are joined by an edge. Formulate the problem of finding the maximal independent set using Integer Linear Programming.

Problem 3 (Matching)**.** A set of edges M of a graph (V, E) is called *matching* if every vertex is covered by at most one edge of M. A matching is *perfect* if every vertex is covered by exactly one edge of M. Using Integer Linear Programming formulate the following two problems.

1. Find the minimal-weight perfect matching.
2. Find the maximal-weight matching.

A weight of matching M is the sum of weights $w(e)$ over all edge $e \in M$.

Problem 4 (Connectivity)**.** Formulate using Linear Programming the decision problem whether a graph is connected.

Problem 5 (Knapsack)**.** Given a set of items, each with a weight and a value, determine the number of each item to include in a collection so that the total weight is less than or equal to a given limit and the total value is as large as possible. Formulate this problem using Integer Linear Programming.

Problem 6 (3-partition)**.** The problem is to decide whether a given multiset of integers can be partitioned into triples that all have the same sum. Formulate this problem using Integer Linear Programming.

Problem 7 (Homework A, 1 point)**.** Write the following problems both in the canonical and the equation forms.

$$\begin{array}{ll} \text{Maximize} & 2\boldsymbol{x}_1 - 3\boldsymbol{x}_2 \\ \text{subject to} & 4\boldsymbol{x}_1 - 5\boldsymbol{x}_2 \leq 6 \\ & 7\boldsymbol{x}_1 + 8\boldsymbol{x}_2 = 8 \\ & \boldsymbol{x}_1 \geq 0. \end{array}$$

$$\begin{array}{ll} \text{Maximize} & \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \\ \text{subject to} & A'\boldsymbol{x} \geq \boldsymbol{b}' \\ & A''\boldsymbol{x} = \boldsymbol{b}'' \\ & \boldsymbol{x} \in \mathbb{R}^n,\ \boldsymbol{x} \geq 0 \end{array}$$

where$A' \in \mathbb{R}^{m' \times n}$, $A'' \in \mathbb{R}^{m'' \times n}$, $\boldsymbol{b}' \in \mathbb{R}^{m'}$, $\boldsymbol{b}'' \in \mathbb{R}^{m''}$, $\boldsymbol{c} \in \mathbb{R}^n$.

Problem 8. Using the graphical methods find the optimal solutions of two objective functions

- $\min \boldsymbol{x}_1 + \boldsymbol{x}_2$
- $\max \boldsymbol{x}_1 + \boldsymbol{x}_2$

subject to the following contitions.

$$\begin{pmatrix} 1 & 3 \\ 1 & 0 \\ 3 & -1 \\ -2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{pmatrix} \geq \begin{pmatrix} 14 \\ 0 \\ 0 \\ -7 \\ 8 \end{pmatrix}$$

Problem 9. Prove that if $A \subseteq \mathbb{R}^n$ is an affine space, then $A - \boldsymbol{x}$ is a linear space for for every $\boldsymbol{x} \in A$. Furthermore, all spaces $A - \boldsymbol{x}$ are the same for all $\boldsymbol{x} \in A$.

Problem 10. Prove that an affine space $A \subseteq \mathbb{R}^n$ is linear if and only if A contains the origin $\mathbf{0}$.

Problem 11 (Homework B, 1 point)**.** Prove that a set $A \subseteq \mathbb{R}^n$ is affine if and only if for every pair of points of A the line defined by those two points is contained in A.

Problem 12. Prove that the set of all solutions of $A\boldsymbol{x} = \boldsymbol{b}$ is an affine space and every affine space is the set of all solutions of $A\boldsymbol{x} = \boldsymbol{b}$ for some A and $\boldsymbol{b}$, assuming $A\boldsymbol{x} = \boldsymbol{b}$ is consistent.

Problem 13. Prove that for vectors $\boldsymbol{v}_0, \ldots, \boldsymbol{v}_k \in \mathbb{R}^n$ the following statements are equivalent.

- Vectors $\boldsymbol{v}_0, \ldots, \boldsymbol{v}_k$ are affinely independent.
- Vectors $\boldsymbol{v}_1 - \boldsymbol{v}_0, \ldots, \boldsymbol{v}_k - \boldsymbol{v}_0$ are linearly independent.
- The origin $\mathbf{0}$ is not a non-trivial combination $\sum \alpha_i \boldsymbol{v}_i$ such that $\sum \alpha_i = 0$ and $\boldsymbol{\alpha} \neq \mathbf{0}$.