Deadline for homeworks: 10.3.2016 at 9:00.

Problem 1. Using the graphical methods find the optimal solutions of two objective functions

- $\min \boldsymbol{x}_1 + \boldsymbol{x}_2$
- $\max \boldsymbol{x}_1 + \boldsymbol{x}_2$

subject to the following contitions.

$$\begin{pmatrix} 1 & 3 \\ 1 & 0 \\ 3 & -1 \\ -2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{pmatrix} \geq \begin{pmatrix} 14 \\ 0 \\ 0 \\ -7 \\ 8 \end{pmatrix}$$

Problem 2 (The Classic Transportation Problem I)**.** An unknown country has n bakeries and m shops. Every day, the i-th bakery bakes b_i breads and the j-the shop sells s_j breads. The transportation of one bread from the i-th bakery to the j-th shop cost $c_{i,j}$. Describe the following problem using linear programming: Find the cheapest transportation of all breads so that all breads are transported according to the number of baked and sold breads.

Furthermore, find necessary and sufficient conditions for existence a feasible and an optimal solutions.

Problem 3 (The Classic Transportation Problem II)**.** It turned out that whenever i-th bakery supplies j-th shop (with positive number of breads), logistic is needed to organize the transportation which cost an extra $l_{i,j}$. Using linear programming formulate the problem of minimizing the total transportation and logistic expenses.

Problem 4. Bandits stole expensive paintings of prices $p_1, \dots, p_n$. Since these bandits were righteous, they wanted to share all paintings out in a fair way. However as they figured out, it was not possible to share them out so that all bandit obtains painting of the same total value. Therefore, they wanted to find the most fair way to share all paintings out.

But how to define fairness? One option is that nobody obtain too much (that is, the most valued share is minimized). Other option is that differences are minimal (that is, minimize the different between the most valued and the least valued share).

How these problems can be modelled using linear programming?

Problem 5 (Homework A: 1 point)**.** The problem is to find the distance between two polyhedrons

$$P^1 = \left\{\boldsymbol{x}^1 \in \mathbb{R}^n;\ A^1\boldsymbol{x}^1 \leq \boldsymbol{b}^1\right\} \text{ and } P^2 = \left\{\boldsymbol{x}^2 \in \mathbb{R}^n;\ A^2\boldsymbol{x}^2 \leq \boldsymbol{b}^2\right\}$$

in the Postman (L_1) metric. The distance of two sets $P_1, P_2 \in \mathbb{R}^n$ is the distance of two closest points $\boldsymbol{x}^1 \in P^1$ and $\boldsymbol{x}^2 \in P^2$. The distance of two points $\boldsymbol{x}^1$ and $\boldsymbol{x}^2$ in the L_1 metric is $\sum_{i=1}^n |\boldsymbol{x}_i^1 - \boldsymbol{x}_i^2|$. Write this problem both in the canonical and the equation forms.

Problem 6. Suppose we have a system of linear inequalities that also contains sharp inequalities. One that may look like this:

$$\begin{array}{rcrcrcrcr} 5x & + & 3y & & & & & \leq & 8 \\ 2x & & & - & 5z & & & < & -3 \\ 6x & + & 5y & & & + & 2w & = & 5 \\ & & & & 3z & + & 2w & > & 5 \\ & & & & & & x,y,z,w & \geq & 0 \end{array}$$

Is there a way to check if this system has a feasible solution using a linear program?

Does this mean that linear programming allows strict inequalities? Not really. As a strange example, construct a “linear program with a strict ineqality” that satisfies the following:

- There is a simple finite upper bound on its optimum value;

- there is a feasible solution; and
- there is no optimal solution.

This may not happen for a linear program – for a bounded LP, once there exists a feasible solution, there exists also an optimal solution.

Problem 7. Prove that if $A \subseteq \mathbb{R}^n$ is an affine space, then $A - \boldsymbol{x}$ is a linear space for for every $\boldsymbol{x} \in A$. Furthermore, all spaces $A - \boldsymbol{x}$ are the same for all $\boldsymbol{x} \in A$.

Problem 8. Prove that an affine space $A \subseteq \mathbb{R}^n$ is linear if and only if A contains the origin $\mathbf{0}$.

Problem 9. Prove that the set of all solutions of $A\boldsymbol{x} = \boldsymbol{b}$ is an affine space and every affine space is the set of all solutions of $A\boldsymbol{x} = \boldsymbol{b}$ for some A and $\boldsymbol{b}$, assuming $A\boldsymbol{x} = \boldsymbol{b}$ is consistent.

Problem 10. Prove that the intersection of arbitrary many convex sets is also a convex set.

Problem 11. Prove that a set $S \subseteq \mathbb{R}^n$ is convex if and only if $S = \operatorname{conv}(S)$.

Problem 12. Prove that the affine hull of a set $S \subseteq \mathbb{R}^n$ is the set of all affine combinations of S.

Problem 13. Prove that all affine bases of an affine space have the same cardinality.

Problem 14. Let S be a linear space and $B \subseteq S \setminus \{\mathbf{0}\}$. Then, B is a linear base of S if and only if $B \cup \{\mathbf{0}\}$ is an affine base of S.

Problem 15 (Homework B: 2 points)**.** Prove that for vectors $\boldsymbol{v}_0, \ldots, \boldsymbol{v}_k \in \mathbb{R}^n$ the following statements are equivalent.

- Vectors $\boldsymbol{v}_0, \ldots, \boldsymbol{v}_k$ are affinely independent.
- Vectors $\boldsymbol{v}_1 - \boldsymbol{v}_0, \ldots, \boldsymbol{v}_k - \boldsymbol{v}_0$ are linearly independent.
- The origin $\mathbf{0}$ is not a non-trivial combination $\sum \alpha_i \boldsymbol{v}_i$ such that $\sum \alpha_i = 0$ and $\boldsymbol{\alpha} \neq \mathbf{0}$.