Deadline for homeworks: 17.3.2016 at 9:00.

Problem 1 (Sudoku)**.** Sudoku can be easily solved using a backtrack. Is it also possible to solve it using Linear programming?

Problem 2 (The shortest path in a graph)**.** Given a weighted graph (V, E, f) where $f : E \to \mathbb{R}^+$, formulate the problem of finding the shortest path from a starting vertex to all other ones using linear programming.

Problem 3. Formulate the Travelling salesman problem using (integer) linear programming.

Problem 4. Plan a production of chocolate for the next year so that the total cost is minimal. The predicted demand of chocolate during the i-th month is d_i units. The change of the production between two consecutive month cost 1500 CZK per unit. Storing chocolate from one month to the following one cost 600 CZK per unit. Chocolate can be stored at most one month because shelf life. As usually, formulate this problem using linear programming.

Is it necessary to consider the production cost?

Problem 5 (Homework A – 2 points)**.** Write a linear programming problem which finds a ball with the maximal radius which is inside a given polyhedron. Properly prove that the optimal solution already gives a ball with the maximal radius.

Problem 6. Prove that the set of all solutions of $A\boldsymbol{x} = \boldsymbol{b}$ is an affine space and every affine space is the set of all solutions of $A\boldsymbol{x} = \boldsymbol{b}$ for some A and $\boldsymbol{b}$, assuming $A\boldsymbol{x} = \boldsymbol{b}$ is consistent.

Problem 7. Prove that a set $S \subseteq \mathbb{R}^n$ is convex if and only if $S = \operatorname{conv}(S)$.

Problem 8. Prove that the affine hull of a set $S \subseteq \mathbb{R}^n$ is the set of all affine combinations of S.

Problem 9. Prove that all affine bases of an affine space have the same cardinality.

Problem 10. Let S be a linear space and $B \subseteq S \setminus \{\mathbf{0}\}$. Then, B is a linear base of S if and only if $B \cup \{\mathbf{0}\}$ is an affine base of S.

Problem 11 (Homework B – 1 point)**.** Let $P = \{x;\ A\boldsymbol{x} \le \boldsymbol{b}\}$ of dimension d. Then for every row i, either

- $P \cap \{x;\ A_{i,\star}\boldsymbol{x} = \boldsymbol{b}_i\} = P$ or
- $P \cap \{x;\ A_{i,\star}\boldsymbol{x} = \boldsymbol{b}_i\} = \emptyset$ or
- $P \cap \{x;\ A_{i,\star}\boldsymbol{x} = \boldsymbol{b}_i\}$ is a proper face of dimension at most $d - 1$.

Problem 12. The intersection of two faces of a polyhedron P is a face of P.

Problem 13. Let P be a polyhedron and $\min \left\{\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x};\ \boldsymbol{x} \in P\right\}$ be a linear programming problem. Then, the set of all optimal solutions is a face of P.

Problem 14 (Extract from Minkovski-Weyl theorem)**.** Let $V \subseteq \mathbb{R}^n$ be a finite set of points (vertices of a polytope). Let

- $Q_2 = \left\{\binom{\boldsymbol{\alpha}}{\beta};\ \boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{v} \le \beta\ \forall \boldsymbol{v} \in V\right\}$,
- $Q_1 = \left\{\binom{\boldsymbol{\alpha}}{\beta};\ \boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{v} \le \beta\ \forall \boldsymbol{v} \in \operatorname{conv}(V)\right\}$,
- $Q = \left\{\binom{\boldsymbol{\alpha}}{\beta};\ \boldsymbol{\alpha} \in \mathbb{R}^n, \beta \in \mathbb{R}, -\mathbf{1} \le \boldsymbol{\alpha} \le 1, -1 \le \beta \le 1, \boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{v} \le \beta\ \forall \boldsymbol{v} \in V\right\}$ and
- W be the set of all vertices of the polytope Q.

Prove the following statements.

1. Let $\boldsymbol{\alpha} \in \mathbb{R}^n$ and $\beta \in \mathbb{R}$. Prove that if for every $v \in V$ holds $\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{v} \le \beta$, then $\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{v} \le \beta$ also holds for every $v \in \operatorname{conv}(V)$. Prove that $Q_1 = Q_2$ follows.
2. Prove that a point $\boldsymbol{x} \in \mathbb{R}^n$ satisfies $\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{x} \le \beta$ for every $\binom{\boldsymbol{\alpha}}{\beta} \in Q_2$ if and only if $\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{x} \le \beta$ holds for every $\binom{\boldsymbol{\alpha}}{\beta} \in Q$.
3. Prove that a point $\boldsymbol{x} \in \mathbb{R}^n$ satisfies $\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{x} \le \beta$ for every $\binom{\boldsymbol{\alpha}}{\beta} \in Q$ if and only if $\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{x} \le \beta$ holds for every $\binom{\boldsymbol{\alpha}}{\beta} \in W$.