Deadline for homeworks: 24.3.2016 at 9:00.

Problem 1 (Homework A – 1 point)**.** A factory has an order to produce n products $J_1, \dots, J_n$. Every product can be produced on one of m machines and the production time in unitary (e.g. one day). Every product i has a release time r_i (i.e. a day when all necessary components are available) and deadline d_i (i.e. the latest production day). The factory's goal is maximizing profit (i.e. number of products made in the planning horizon). Model this problem using (Integer) Linear Programming.

Problem 2 (Extract from Minkovski-Weyl theorem, Homework B – 2 points)**.** Let $V \subseteq \mathbb{R}^n$ be a finite set of points (vertices of a polytope). Let

- $Q_2 = \left\{ \binom{\boldsymbol{\alpha}}{\beta};\ \boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{v} \le \beta\ \forall \boldsymbol{v} \in V \right\}$,
- $Q_1 = \left\{ \binom{\boldsymbol{\alpha}}{\beta};\ \boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{v} \le \beta\ \forall \boldsymbol{v} \in \operatorname{conv}(V) \right\}$,
- $Q = \left\{ \binom{\boldsymbol{\alpha}}{\beta};\ \boldsymbol{\alpha} \in \mathbb{R}^n, \beta \in \mathbb{R}, -\mathbf{1} \le \boldsymbol{\alpha} \le 1, -1 \le \beta \le 1, \boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{v} \le \beta\ \forall \boldsymbol{v} \in V \right\}$ and
- W be the set of all vertices of the polytope Q.

Prove the following statements.

1. Let $\boldsymbol{\alpha} \in \mathbb{R}^n$ and $\beta \in \mathbb{R}$. Prove that if for every $v \in V$ holds $\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{v} \le \beta$, then $\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{v} \le \beta$ also holds for every $v \in \operatorname{conv}(V)$. Prove that $Q_1 = Q_2$ follows.
2. Prove that a point $\boldsymbol{x} \in \mathbb{R}^n$ satisfies $\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{x} \le \beta$ for every $\binom{\boldsymbol{\alpha}}{\beta} \in Q_2$ if and only if $\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{x} \le \beta$ holds for every $\binom{\boldsymbol{\alpha}}{\beta} \in Q$.
3. Prove that a point $\boldsymbol{x} \in \mathbb{R}^n$ satisfies $\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{x} \le \beta$ for every $\binom{\boldsymbol{\alpha}}{\beta} \in Q$ if and only if $\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{x} \le \beta$ holds for every $\binom{\boldsymbol{\alpha}}{\beta} \in W$.

Problem 3. Let n-dimensional hypercube be the set $\{\boldsymbol{x} \in \mathbb{R}^n;\ \mathbf{0} \le \boldsymbol{x} \le \mathbf{1}\}$. Let n-dimensional simplex be the convex hull of $n+1$ affinely independent points such that no point belongs into the convex hull of the other points. Determine the number of k-dimensional faces of the n-dimensional hypercube and the n-dimensional simplex.

Problem 4. Prove that the n-dimensional ball is not a polyhedron.

Problem 5. The intersection of two faces of a polyhedron P is a face of P.

Problem 6. Prove that the set of all solutions of $A\boldsymbol{x} = \boldsymbol{b}$ is an affine space and every affine space is the set of all solutions of $A\boldsymbol{x} = \boldsymbol{b}$ for some A and $\boldsymbol{b}$, assuming $A\boldsymbol{x} = \boldsymbol{b}$ is consistent.

Problem 7. Prove that a set $S \subseteq \mathbb{R}^n$ is convex if and only if $S = \operatorname{conv}(S)$.

Problem 8. Prove that the affine hull of a set $S \subseteq \mathbb{R}^n$ is the set of all affine combinations of S.

Problem 9. Prove that all affine bases of an affine space have the same cardinality.

Problem 10. Let S be a linear space and $B \subseteq S \setminus \{\mathbf{0}\}$. Then, B is a linear base of S if and only if $B \cup \{\mathbf{0}\}$ is an affine base of S.