Problem 1.

$$\begin{aligned} -x_1+2x_2+x_3+4x_4 &= 3\\ -2x_1+4x_2+x_3+7x_4 &= 5\\ x_1-2x_2+x_3-2x_4 &= -1\\ -x_1+2x_2+2x_3+5x_4 &= 4 \end{aligned}$$

Problem 2. Find all vertices of a polyhedron determined by the following conditions.

$$\begin{array}{rcrcrcl} x_1 & + & x_2 & + & x_3 & \leq & 8\\ x_1 & & & & & \leq & 6\\ & & x_2 & + & x_3 & \leq & 4\\ & & & & x_1,x_2,x_3 & \geq & 0 \end{array}$$

Problem 3 (Homework – 1 point)**.** Find all vertices of a polyhedron determined by the following conditions.

$$\begin{array}{rcrcrcr} 2x_1 & + & x_2 & + & x_3 & \leq & 14\\ 2x_1 & + & 5x_2 & + & 5x_3 & \leq & 30\\ & & & & x_1,x_2,x_3 & \geq & 0 \end{array}$$

Problem 4 (Homework – 1 point)**.** The convex hull of points $(0,1,0,1,0), (0,1,0,\frac{10}{11},\frac{10}{11})$, $(0,0,1,1,0)$, $(0,0,1,\frac{10}{11},\frac{10}{11})$ is a face F of a polyherdon P given by conditions

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & + & x_3 & \leq & 1\\ & & x_4 & + & 10x_5 & \leq & 10\\ & & 10x_4 & + & x_5 & \leq & 10\\ & & & & x_1,x_2,x_3,x_4,x_5 & \geq & 0. \end{array}$$

Find an objective function $c^T x$ such that the set of all optimal solution of the linear problem $\max\left\{c^T x;\ x \in P\right\}$ is exactly F. Prove that your objective function already gives the face F.

Problem 5. Prove that the system of linear equation $A\boldsymbol{x} = \boldsymbol{b}$ has a solution if and only if the system $\boldsymbol{y}^{\mathrm{T}} A = 0$ and $\boldsymbol{y}^{\mathrm{T}} b = -1$ has no solution.

Problem 6. Prove that the set of faces of a polyhedron and the inclusion form a partially ordered set.

Problem 7. Prove that every polyerd $P = \{\boldsymbol{x} \in \mathbb{R}^n;\ A\boldsymbol{x} \leq \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\}$ has a vertex.

Problem 8. Prove that every proper (inclusion) maximal face is a facet.

Problem 9. Prove that every polytope has a vertex.

Problem 10. Prove that all proper (inclusion) minimal faces have the same dimension.

Problem 11. Every proper face is an intersection of facets.