Problem 1. Write a linear programming program which decides whether a given oriented graph contains an oriented cycle.

Problem 2. Solve the following problem

$$\begin{array}{lrcrcr} \text{Maximize} & 3x_1 & + & x_2 & & \\ \text{subject to} & x_1 & - & x_2 & \leq & -1 \\ & -x_1 & - & x_2 & \leq & -3 \\ & 2x_1 & + & x_2 & \leq & 2 \\ & & & x_1, x_2 & \geq & 0 \end{array}$$

Problem 3. Solve the following problem

$$\begin{array}{lrcrcr} \text{Maximize} & 3x_1 & + & x_2 & & \\ \text{subject to} & x_1 & - & x_2 & \leq & -1 \\ & -x_1 & - & x_2 & \leq & -3 \\ & 2x_1 & - & x_2 & \leq & 2 \\ & & & x_1, x_2 & \geq & 0 \end{array}$$

Problem 4. Without using the Farkas' lemma, prove that the system of linear equation $A\boldsymbol{x} = \boldsymbol{b}$ has a solution if and only if the system $\boldsymbol{y}^{\mathrm{T}}A = 0$ and $\boldsymbol{y}^{\mathrm{T}}b = -1$ has no solution.

Problem 5. Find the dual problem to the following linear programming problems and write the complementary slackness conditions.

1. $\max c^T x$ subject to $Ax \leq b$,
2. $\max c^T x$ subject to $Ax = b, x \geq 0$,
3. $\min c^T x$ subject to $A_1 x = b_1, A_2 x \geq b_2$.

Problem 6. Find the dual problem to the following linear programming problem and write the complementary slackness conditions.

$$\begin{array}{lrcrcrcr} \text{Minimize} & 2x_1 & & & - & 2x_3 & & \\ \text{subject to} & 7x_1 & + & 10x_2 & + & 2x_3 & \leq & 23 \\ & 2x_1 & + & 3x_2 & + & x_3 & = & 5 \\ & -4x_1 & + & 14x_2 & - & 3x_3 & \geq & 11 \\ & & & & & x_3 & \geq & 0 \\ & & & & & x_3 & \leq & 0 \end{array}$$

Problem 7 (Homework 2 points)**.** Find the dual problem to the following two linear programming problems and write the complementary slackness conditions.

$$\begin{array}{rrcrcrcrcr} \text{1) Maximize} & x_1 & - & 2x_2 & & & + & 3x_4 & & \\ \text{2) Minimize} & x_1 & - & 2x_2 & & & + & 3x_4 & & \\ \text{subject to} & & & x_2 & - & 6x_3 & + & x_4 & \leq & 4 \\ & -x_1 & + & 3x_2 & - & 3x_3 & & & = & 0 \\ & 6x_1 & - & 2x_2 & + & 2x_3 & - & 4x_4 & \geq & 5 \\ & \multicolumn{9}{l}{x_2 \leq 0,\ x_4 \geq 0} \end{array}$$

Problem 8 (Homework, 2 points)**.** Let $A \in R^{m \times n}$ and $\boldsymbol{b} \in \mathbb{R}^m$. Prove that

- the system $A\boldsymbol{x} = \boldsymbol{b}$ has a non-negative solution $\boldsymbol{x} \in \mathbb{R}^n$ if and only if every $\boldsymbol{y} \in \mathbb{R}^m$ with $\boldsymbol{y}^{\mathrm{T}}A \geq \boldsymbol{0}^{\mathrm{T}}$ satisfies $\boldsymbol{y}^{\mathrm{T}}\boldsymbol{b} \geq 0$,
- the system $A\boldsymbol{x} \leq \boldsymbol{b}$ is infeasible if and only if $\boldsymbol{0x} \leq -1$ is a non-negative linear combination of inequalities $A\boldsymbol{x} \leq \boldsymbol{b}$.