Problem 1. Without using the Farkas' lemma, prove that the system of linear equation $A\boldsymbol{x} = \boldsymbol{b}$ has a solution if and only if the system $\boldsymbol{y}^{\mathrm{T}}A = 0$ and $\boldsymbol{y}^{\mathrm{T}}b = -1$ has no solution.

Problem 2. Find the dual problem to the following linear programming problems and write the complementary slackness conditions.

1. $\max c^T x$ subject to $Ax \leq b$,
2. $\max c^T x$ subject to $Ax = b, x \geq 0$,
3. $\min c^T x$ subject to $A_1 x = b_1, A_2 x \geq b_2$.

Problem 3. Find the dual problem to the following linear programming problem and write the complementary slackness conditions.

$$\begin{array}{lrcrcrcr}
\text{Minimize} & 2x_1 & & & - & 2x_3 & & \\
\text{subject to} & 7x_1 & + & 10x_2 & + & 2x_3 & \leq & 23 \\
& 2x_1 & + & 3x_2 & + & x_3 & = & 5 \\
& -4x_1 & + & 14x_2 & - & 3x_3 & \geq & 11 \\
& & & & & x_3 & \geq & 0 \\
& & & & & x_3 & \leq & 0
\end{array}$$

Problem 4. Formulate the minimal s-t cut problem in an oriented graph $G = (V, E)$ using linear programming. First, find conditions on variables $y_u, z_{uv} \in \{0, 1\}$ for $u \in V$ and $uv \in E$ such that

- $U = \{u \in V;\ x_u = 1\}$ and $V \setminus U = \{u \in V;\ x_u = 0\}$ is a partitioning of vertices into two disjoint subsets,
- $s \in U$ and $t \notin U$ and
- $z_{uv} = 1$ if and only if $u \in U$ and $v \notin U$.

Then, find an objective function which minimize the weight of a cut. Finally, try to minimize the number of inequalities and remove all integral condition.

Problem 5. Prove that A is totally unimodular if and only if $(A|I)$ is unimodular.

Problem 6. Let $P = \operatorname{conv}\left\{(0,0), (1,0), (\frac{1}{2}, 3)\right\}$ and P' be the convex hull of all integral points of P. First, find a system of linear inequalities which determines P. Then, using Chvátal-Gomory cutting planes derive a system of linear inequalities which determines P'.

Problem 7 (Homework)**.** Let $P = \operatorname{conv}\left\{(0,0), (1,0), (\frac{1}{2}, k)\right\}$ where $k \in \mathbb{N}$ and P' be the convex hull of all integral points of P. First, find a system of linear inequalities which determines P. Then, describe a sequence of Chvátal-Gomory cutting planes which leads to a system of linear inequalities which determines P'. How many cutting planes do you need?

Problem 8. Prove that the incidence matrix A of a graph G is totally unimodular if and only if G is bipartite.

Problem 9. Find a 0–1 matrix A and an integral vector $\boldsymbol{b}$ such that $\{\boldsymbol{x};\ A\boldsymbol{x} \leq \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\}$ is an integral polytope but A is not totally unimodular.

Problem 10 (Homework)**.** Let M be a matching of G and let p be the cardinality of the maximum matching. Prove that there are at least $p - |M|$ vertex-disjoint M-augmenting paths.