Remainer: The deadline for the second practical homework is 8.5.

Problem 1. Prove that A is totally unimodular if and only if $(A|I)$ is unimodular.

Problem 2. Prove that the incidence matrix A of a graph G is totally unimodular if and only if G is bipartite.

Problem 3. Find a 0–1 matrix A and an integral vector $\boldsymbol{b}$ such that $\{\boldsymbol{x};\ A\boldsymbol{x} \leq \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\}$ is an integral polytope but A is not totally unimodular.

Problem 4 (Homework)**.** Consider the linear programming for Minimum-Weight perfect matchings in general graphs:

$$\begin{array}{lrcll} \text{Minimize} & \boldsymbol{cx} & & & \\ \text{subject to} & \delta^u \boldsymbol{x} & = & 1 & \text{for all } u \in V \\ & \delta^D \boldsymbol{x} & \geq & 1 & \text{for all } D \in \mathcal{C} \\ & \boldsymbol{x} & \geq & \boldsymbol{0} & \end{array}$$

Where $\delta^D \in \{0,1\}^E$ is a vector such that $\delta^D_{uv} = 1$ if $|uv \cap D| = 1$ and $\delta^w = \delta^{\{w\}}$ and $\mathcal{C}$ is the set of all odd-size subsets of V.

From conditions $\delta^u \boldsymbol{x} = 1$ and $\boldsymbol{x} \geq \boldsymbol{0}$ derive using Gomory-Chvátal cutting planes inequalities $\delta^D \boldsymbol{x} \geq 1$.

Problem 5. Prove that the linear programming

$$\begin{array}{lrcll} \text{Minimize} & \boldsymbol{cx} & & & \\ \text{subject to} & \delta^u \boldsymbol{x} & = & 1 & \text{for all } u \in V \\ & \delta^D \boldsymbol{x} & \geq & 1 & \text{for all } D \in \mathcal{C} \\ & \boldsymbol{x} & \geq & \boldsymbol{0} & \end{array}$$

is feasible if and only if G has a perfect matching (without using algorithms from the lecture). Also prove the convex hull of characteristic vectors of perfect matchings is exactly the set of all feasible solution this set of linear inequalities.

Problem 6 (Homework)**.** Let $\operatorname{def}(G)$ be the number of exposed edges by a maximal-size matching in $G = (V, E)$. Find a graph G with odd circuit C such that $\operatorname{def}(G) < \operatorname{def}(G \times C)$.

Problem 7. For every $n \geq 3$ find a connected graph on n vertices such that the relaxed linear programming problem for perfect matching has feasible solution.

Problem 8. Let G be a bipartite graph and $\boldsymbol{x}$ be relaxed feasible solution of the linear programming problem for perfect matching. If $\boldsymbol{x}$ is not integral, then prove that$G$ contains a cycle C such that $\boldsymbol{x}_e$ is non-integral for every edge e on the cycle C. Find how the the cycle C can be used to reduce the number of non-integral values of $\boldsymbol{x}$. Is the assumption that G is bipartite necessary?

Problem 9. Slither is a two-person game played on a graph $G = (V, E)$. The players play alternatively. At each step the player whose turn it is chooses a previously unchosen edge. The only rule is that at every step the set of chosen edges forms a path. The loser is the player unable to extern the path. Prove that, if G has a perfect matching, then the first player has a winning strategy.

Problem 10.