Hypercube problems

Homework Assignment 3

Due: Jan 4, 2021 Updated: December 6, 2020

1. Show that every sequence (a_1, a_2, a_3, a_4) of even nonnegative integers with $\sum_i a_i = 2^3$ is a projection vector (spectrum) of some perfect matching in Q_4.

2. Show that every induced matching in a hypercube extends to a perfect matching. A matching is *induced* if it is induced by its vertex set; that is, vertices of distinct edges are nonadjacent.

3. Show that the maximal size of an induced matching in Q_n is 2^{n-2}.

4. A cyclic Gray code Γ in Q_n (i.e. a Hamiltonian cycle) is *complementary* if every two antipodal vertices of Q_n are antipodal also in Γ (that is, they appear 2^{n-1} steps from each other). Show that Q_n has a complementary code for every even $n \geq 2$.

5. Recall a reflected Gray code: $\Gamma_1 = (0, 1)$, $\Gamma_n = (0\Gamma_{n-1}, 1\Gamma^R_{n-1})$ for $n \geq 2$. Show that for $\Gamma_n = (v_0, \ldots, v_{2^n-1})$ of dimension n, for every $0 \leq i \leq 2^n - 1$,

$$v_i = (i)_2 \oplus (\lfloor i/2 \rfloor)_2,$$

$$(i)_2 = v_i \oplus \mathtt{sh}_0(v_i) \oplus \mathtt{sh}_0^2(v_i) \oplus \ldots \oplus \mathtt{sh}_0^{n-1}(v_i)$$

   where $(i)_2$ is the standard binary repr. of i and $\mathtt{sh}_0(x_n \ldots x_1) = (0x_n \ldots x_2)$ (shift).

6. Show that the binomial tree B_n has a total distance (*Wiener index*)

$$td(B_n) = 2td(Q_n) - \binom{2^n}{2}.$$

7. Show that for every $x, y \in \{-1, 1\}^n$ and every $S, T \subseteq [n]$

   - $\chi_S(x \oplus y) = \chi_S(x) \cdot \chi_S(y)$
   - $\chi_{S\Delta T}(x) = \chi_S(x) \cdot \chi_T(x)$

   where $x \oplus y = (x_1y_1, \ldots, x_ny_n)$, Δ denotes the symmetric difference, and

$$\chi_S(x) = \prod_{i \in S} x_i \quad \text{if } S \neq \emptyset, \text{ and } \chi_\emptyset(x) = 1.$$

8. Applying the above equalities show that $\{\chi_S\}_{S \subseteq [n]}$ is an orthogonal basis of the space Ω_n of all functions $f : \{-1, 1\}^n \to \mathbb{R}$.