Hypercube problems

Homework Assignment 3

Due: Jan 6, 2021 Updated: December 2, 2021

1. Find a boolean term (with $\neg, \wedge, \vee$) that computes the median $med(x, y, z)$ of x, y, z (in the boolean algebra $\underline{2}^{[n]}$).
2. Show that every induced matching in a hypercube extends to a perfect matching. A matching is *induced* if it is induced by its vertex set; that is, vertices of distinct edges are nonadjacent.
3. Determine what is the maximal size of an induced matching in Q_n. Find a direct proof (not using semi-induced matching).
4. Show that for every $x, y \in \{-1, 1\}^n$ and every $S, T \subseteq [n]$
   - $\chi_S(x \oplus y) = \chi_S(x) \cdot \chi_S(y)$
   - $\chi_{S \Delta T}(x) = \chi_S(x) \cdot \chi_T(x)$

   where $x \oplus y = (x_1 y_1, \ldots, x_n y_n)$, Δ denotes the symmetric difference, and
   $$\chi_S(x) = \prod_{i \in S} x_i \quad \text{if } S \neq \emptyset, \text{ and } \chi_\emptyset(x) = 1.$$
5. Applying the above equalities show that $\{\chi_S\}_{S \subseteq [n]}$ is an orthogonal basis of the space Ω_n of all functions $f : \{-1, 1\}^n \to \mathbb{R}$.
6. Let $n = k^2$ where k is even and let $f : \{0, 1\}^n \to \{0, 1\}$ be the (Rubinstein's) function defined by $f(x_{11}, \ldots, x_{1k}, \ldots, x_{k1}, \ldots, x_{kk}) = 1$ if and only if there is some $1 \leq j \leq k$ such that $x_{j1}, \ldots, x_{jk}$ contain exactly two consecutive 1s starting at odd position and the other values are 0s. Show that f has sensitivity $s(f) = \sqrt{n}$ and block sensitivity $bs(f) = n/2$.
7. A cyclic Gray code Γ in Q_n (i.e. a Hamiltonian cycle) is *complementary* if every two antipodal vertices of Q_n are antipodal also in Γ (that is, they appear 2^{n-1} steps from each other). Show that Q_n has a complementary code for every even $n \geq 2$.
8. Recall a reflected Gray code: $\Gamma_1 = (0, 1)$, $\Gamma_n = (0\Gamma_{n-1}, 1\Gamma_{n-1}^R)$ for $n \geq 2$. Show that for $\Gamma_n = (v_0, \ldots, v_{2^n-1})$ of dimension n, for every $0 \leq i \leq 2^n - 1$,
   $$v_i = (i)_2 \oplus (\lfloor i/2 \rfloor)_2,$$
   $$(i)_2 = v_i \oplus \mathtt{sh}_0(v_i) \oplus \mathtt{sh}_0^2(v_i) \oplus \ldots \oplus \mathtt{sh}_0^{n-1}(v_i)$$
   where $(i)_2$ is the standard binary repr. of i and $\mathtt{sh}_0(x_n \ldots x_1) = (0x_n \ldots x_2)$ (shift).