Hypercube problems

Lecture 1

October 11, 2012

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Preliminaries

For a positive integer n let $[n] = \{1, 2, \dots, n\}$. Let $\oplus$ denote the addition in $\mathbb{Z}_2^n$ and for $i \in [n]$ let e_i denote the vector in $\mathbb{Z}_2^n$ with 1 exactly in the ith coordinate. Let $\mathbf{0}$ and $\mathbf{1}$ denote the vectors of all 0's and 1's, respectively. The *Hamming distance* of $u = (u_1, \dots, u_n)$, $v = (v_1, \dots, v_n) \in \mathbb{Z}_2^n$ is

$$d_H(u, v) = |\{\, i \in [n] \mid u_i \neq v_i \}| \,.$$

Definition 1 *The n-dimensional hypercube Q_n (also known as an n-cube, a Boolean cube, a discrete cube) is an undirected graph with $V(Q_n) = \mathbb{Z}_2^n = \{0,1\}^n$ and*

$$E(Q_n) = \{\, uv \mid u \oplus v = e_i \text{ for some } i \in [n]\} = \{\, uv \mid d_H(u, v) = 1\} \,.$$

If $u \oplus v = e_i$, the edge $uv \in E(Q_n)$ is said to have *direction i*.

Hypercube architectures (an excerpt from history)

1983-87 Cosmic Cube – Caltech ($n = 2, 6, 7$)

1983-87 Connection Machine CM-1, CM-2, CM-200 – MIT[1] ($n = 16, 9, 13$)

1985-90 Intel iPSC/1, iPSC/2, iPSC/860 ($n = 7$)

1986-89 nCUBE-1, nCUBE-2 – nCUBE Corporation ($n = 10, 13$)

1980's other manufacturers: Floating Point Corporation (T series), Ametek

1997 SGI Origin 2000 – partly involves hypercubes [3]

2002 HyperCuP – p2p networks [11]

2006 BlueCube – Bluetooth networks [2]

2011 HyperD – dynamic distributed databases [12]

[1] Richard Feynman, an American theoretical physicist was involved in its design [4].

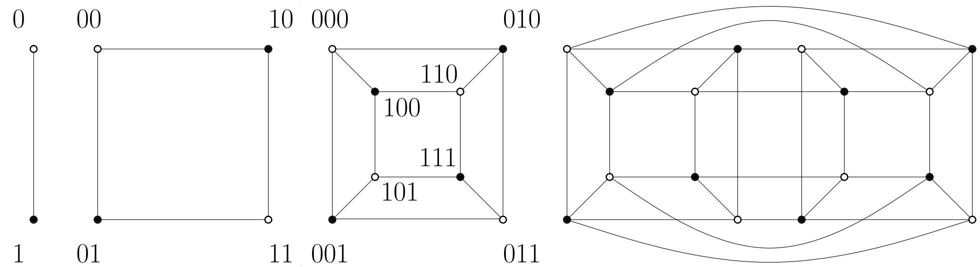

Figure 1: Hypercubes Q_n for $n = 1, 2, 3, 4$.

Alternative definitions of hypercubes

- $Q_n = K_2^n = \underbrace{K_2 \,\square\, K_2 \,\square \cdots \square\, K_2}_{n-\text{times}}$ (if $n \geq 1$) and $Q_0 = K_1$.

  The Cartesian product $G \,\square\, H$ of graphs G and H is the graph with the vertex set $V(G \,\square\, H) = V(G) \times V(H)$ and the edge set
  $$E(G \,\square\, H) = \left\{ (u,v)(u',v) \;\middle|\; uu' \in E(G) \right\} \cup \left\{ (u,v)(u,v') \;\middle|\; vv' \in E(H) \right\}.$$

- Q_n is the covering graph of the *Boolean lattice* $\mathcal{B}_n = (\mathcal{P}(X), \subseteq)$ with $|X| = n$. [2]

  The covering graph of a poset is the graph of its Hasse diagram. For $i \in [n]$ the i-th *level* of $\mathcal{B}_n$ is $X^i = \{ A \subseteq X \mid |A| = i \}$.

- Q_n is the 1-skeleton of the polytope $[0,1]^n$.

  The vertices (edges) are 0-faces (1-faces, resp.) of $[0,1]^n$.

- $Q_n = \mathrm{Cay}(\mathbb{Z}_2^n, \{e_1, \ldots, e_n\})$.

  A *Cayley graph* of a group Γ generated by a set $S \subseteq \Gamma$ that is closed under inverses (to be undirected) and $e \notin S$ (to have no loops) is $\mathrm{Cay}(\Gamma, S) = (\Gamma, \left\{ uv \mid vu^{-1} \in S \right\})$.

Basic properties of hypercubes

For every $n \geq 1$,

- $|V(Q_n)| = 2^n$.
- $|E(Q_n)| = n2^{n-1}$.

  The edges of each direction form a perfect matching. Each direction splits Q_n into two copies of Q_{n-1}. Since Q_n has precisely n directions the number of edges is $n2^{n-1}$.

- Q_n is n-regular.

  A graph is k-regular if every vertex has degree k. The neighborhood of $u \in V(Q_n)$ is $N(u) = \{ u \oplus e_i \mid i \in [n] \}$.

[2] Equivalently, Q_n is the covering graph of the $(n-1)$-simplex face lattice. The *n-simplex* is a convex hull of $n-1$ points in a general position (e.g. 2-simplex is a triangle, 3-simplex is a tetrahedron). The face lattice is formed by faces (together with $\emptyset$ representing the (-1)-dimensional face) ordered by inclusion.

- Q_n is bipartite.

  The *size* of u is defined by $|u| = d_H(u, \mathbf{0}) = |\{ i \in [n] \mid u_i = 1\}|$. This allows us to define *odd (even) vertices* u by their size. They form the bipartition of Q_n.

- $diam(Q_n) = rad(Q_n) = n$.

  The *diameter (radius)* of a graph G is the maximal (minimal, resp.) eccentricity in G. The *eccentricity* of a vertex is its greatest distance to any other vertex.

- Q_n is both n-connected and n-edge-connected.

  The (vertex)-connectivity and the edge connectivity of Q_n is $\kappa(Q_n) = \lambda(Q_n) = n$.

- $g(Q_n) = 4$ and $circ(Q_n) = 2^n$ (i.e. Q_n is Hamiltonian) if $n \geq 2$.

  The *girth* $g(G)$ and the *circumference* $circ(G)$ is the length of the shortest (the longest, resp.) cycle in G.

- An example of a Hamiltonian cycle in Q_n is a *reflected Gray code* Γ_n defined recursively by $\Gamma_1 = (0, 1)$ and $\Gamma_{n+1} = (0\Gamma_n, 1\Gamma_n^R)$ where Γ_n^R is the reflection of Γ_n.

- Q_n is bipancyclic.

  A graph G is (bi)pancyclic if it has cycles of all (even) lengths l for $g(G) \leq l \leq circ(G)$.

- Q_n has $(0, 2)$-property. (We also say that Q_n is a $(0, 2)$-graph.)

  A graph G has $(0, 2)$-*property* if every two vertices have 0 or 2 common neighbors. If x is a common neighbor of u and v, their second common neighbor is $u \oplus v \oplus x$.

- Q_n has $\binom{n}{k} 2^{n-k}$ k-dimensional subcubes and 3^n of all subcubes.

  A k-dimensional subcube is a subgraph isomorphic to Q_k. All subcubes in Q_n are induced subcubes. Let $w \in \{0, 1, *\}^n$ and let k be the number of $*$'s in w. Then w uniquely represents the k-subcube (denoted by $Q_n[w]$) induced by

  $$\{ (v_1, \ldots, v_n) \in \mathbb{Z}_2^n \mid v_i = w_i \text{ if } w_i \neq * \}.$$

  Thus, the number of k-subcubes is the number of $w \in \{0, 1, *\}^n$ with k stars.

- $Q_{n+m} \simeq Q_n \,\square\, Q_m$ for every $m \geq 1$.

  For an m-tuple $I = (i_1, \ldots, i_m)$ where $i_j \in [n]$ are distinct and $w \in \{0, 1\}^m$ we denote

  $$Q_{n+m}^{I,w} = Q_{n+m}[s] \text{ where } s_j = \begin{cases} * & \text{if } j \notin I \\ w_{i_j} & \text{if } j \in I \end{cases}.$$

  That is, $Q_{n+m}^{I,w}$ is the n-dimensional subcube with m coordinates I fixed by w.

The genus of the hypercube

The *orientable genus* $\gamma(G)$ of a graph G is the minimal genus of an orientable surface where G can be embedded without crossing (e.g. $\gamma(G) = 0$ if and only if G is planar).

Theorem 2 (Ringel [10]) *For every* $n \geq 2$, $\gamma(Q_n) = (n-4)2^{n-3} + 1$.

Proof Applying Euler's formula $V - E + F = 2 - 2\gamma(G)$ and the fact that $4F \leq 2E$ if G is triangle-free, we obtain $\gamma(G) \geq \frac{E}{4} - \frac{V}{2} + 1$. Substituting values for Q_n results in $\gamma(Q_n) \geq \frac{n2^{n-1}}{4} - 2^{n-1} + 1 = (n-4)2^{n-3} + 1$. Let us denote this number by γ_n.

Claim 3 *For every* $n \geq 2$, Q_n *has an embedding to* S_{γ_n} *(an orientable surface of genus* γ_n*) such that every subcube* $*w*$ *(of dimension 2) is a face.*

We prove the claim by induction. For $n = 2$ it is obvious. Assuming it holds for $n-1$ we prove it for n. We start with two copies Q^0_{n-1}, Q^1_{n-1} of Q_{n-1}, each embedded to its surface $S^0_{\gamma_{n-1}}$, $S^1_{\gamma_{n-1}}$. We add 0 to each vertex in Q^0_{n-1} and 1 to each vertex in Q^1_{n-1}. For each $w \in \{0,1\}^{n-3}$ we add a handle that connects a face $*w*0$ of Q^0_{n-1} to a face $*w*1$ of Q^1_{n-1}, see Figure 2. The first handle connects $S^0_{\gamma_{n-1}}$ and $S^1_{\gamma_{n-1}}$ into a surface of genus $2\gamma_{n-1}$. Each additional handle increases genus by one. Thus we obtain a surface of genus $2\gamma_{n-1} + 2^{n-3} - 1 = \gamma_n$. Observe on Figure 2 that each subcube $*w*$ is a face in the obtained embedding of Q_n into S_{γ_n}. ■

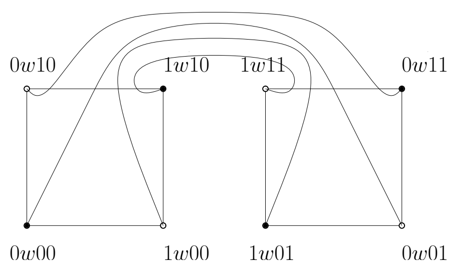

Figure 2: The connection of faces $*w*0$ and $*w*1$.

The *non-orientable genus* $\tilde{\gamma}(G)$ of a graph G is the minimal genus of a non-orientable surface[3] where G can be embedded without crossing.

Theorem 4 (Jungerman [8]) *For every* $n \geq 1$,

$$\tilde{\gamma}(Q_n) = \begin{cases} 1 & \textit{if } n \leq 3, \\ 3 + 2^{n-2}(n-4) & \textit{if } n = 4, 5, \\ 2 + 2^{n-2}(n-4) & \textit{if } n \geq 6. \end{cases}$$

Let us define two following supergraphs of Q_n. The first one has additional edges joining *antipodal* vertices and is also known as a *hypercube with diagonals*.

[3]The *genus of a non-orientable (closed connected) surface* is the number of crosscaps needed to be attached to a sphere to obtain a topologically equivalent surface. A *crosscap* is a circle twisted so that entering at one side results in coming out from the opposite side (e.g. a sphere with one crosscap is the real projective plane, a sphere with two crosscaps is the Klein bottle).

Definition 5 *The folded cube FQ_n of dimension n is a graph with $V(FQ_n) = \mathbb{Z}_2^n$ and*

$$E(FQ_n) = E(Q_n) \cup \{uv \mid u \oplus v = \mathbf{1}\}.$$

The second one has additional edges joining vertices that differ in a *prefix*.

Definition 6 *The augmented cube AQ_n of dimension n is a graph with $V(AQ_n) = \mathbb{Z}_2^n$ and*

$$E(AQ_n) = E(Q_n) \cup \left\{ uv \mid u \oplus v = \sum_{j=1}^{i} e_j \text{ for some } i \in [n] \right\}.$$

Problem 1 *Determine $\gamma(FQ_n)$, $\gamma(AQ_n)$ and $\tilde{\gamma}(FQ_n)$, $\tilde{\gamma}(AQ_n)$.*

Update: It is known [6] that $\gamma(FQ_n) = (n-3)2^{n-3} + 1$ if n is odd and $(n-3)2^{n-3} + 1 \leq \gamma(EQ_n) \leq (n-2)2^{n-3} + 1$ if n is even.

Notes

A nice (but outdated) survey on properties of hypercubes is [7]. Hypercubes as interconnection networks are studied in [3] and [9]. The above proof of Theorem 2 follows [1]. For standard graph terminology see e.g. [5].

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