Hypercube problems

Lecture 26

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1 Gray codes

In a broader context, a combinatorial *Gray code* refers to enumerating all objects of some combinatorial class into a sequence so that consecutive objects differ only by a "small amount". For example enumerating all

- permutations in S_n so that consecutive permutations differ by an adjacent transposition,
- strings, necklaces differing by a flip in a single position,
- k-sets of an n-set, so that consecutive subsets can be obtained by swapping a single element,
- set partitions of a set, where in each step of the enumeration a single element is moved to an adjacent block,
- integer partitions of some natural number, only with increment or decrement by 1 in only two parts, at each step of the enumeration,
- binary trees on n vertices differing by a single tree rotation,
- triangulations of a convex polygon, with consecutive ones differing by a flip of a single diagonal,
- spanning trees for a given graph, such that consecutive spanning trees can be obtained by swapping a single edge,
- acyclic orientations of a given graph, so that consecutive orientations differ only in the orientation of a single edge,
- linear extensions of a poset differing by adjacent transpositions.

COMBOS (The Combinatorial Object Server) is a place one can play around with combinatorial Gray codes for various classes.

There are different algorithms one can hope for a combinatorial Gray code:

- **Loopless**: an algorithm that takes $\mathcal{O}(1)$ per object in enumeration.

- **CAT** (a constant average time): $\mathcal{O}(N)$ time is spend to enumerate all objects, N being the total number of objects.
- **Ranking/Un-ranking**: A ranking algorithm aims to output the index in the enumeration given an object. And un-ranking algorithm computes the inverse function.

In a flip graph the combinatorial objects correspond to vertices and edges are between vertices that correspond to objects obtainable by "small change" (flip). Thus a (cyclic) combinatorial Gray code can also be thought of as a Hamiltonian path (cycle) in the corresponding flip graph. For example,

- for binary strings of length n, with the "small change" being a single bit flip the corresponding flip graph is an n-cube,
- for permutations of $[n]$ w.r.t adjacent transposition (the "small change"), the corresponding flip graph is a permutahedron,
- for binary trees on n vertices w.r.t single tree rotation (the "small change"), the corresponding flip graph is an associahedron.

Remark As an informal remark, many flip graphs are usually highly symmetric, in the sense they are vertex transitive and sometimes Cayley graphs. In this context the following conjecture makes us optimistic about the existence of combinatorial Gray codes for various classes.

Conjecture 1 (Lovász [18]) [1] *Every connected vertex-transitive graph has a Hamilton path.*

Many flip graphs are cover graphs of some posets or skeletons of high dimensional polytopes. For example,

- the hypercube is the cover graph of the Boolean lattice,
- the permutahedron is the cover of the weak (Bruhat) order[2] of S_n,
- the associahedron is the cover graph of the Tamari lattice.

And the above examples can geometrically be seen as polytopes as well.

Definition 2 (Lattice congruence) *A lattice congruence is an equivalence relation $\equiv$ on a lattice P that is compatible with taking joins and meets. Formally, if $\pi \equiv \pi^{'}$ and $\rho \equiv \rho^{'}$ then we also have $\pi \vee \rho \equiv \pi^{'} \vee \rho^{'}$ and $\pi \wedge \rho \equiv \pi^{'} \wedge \rho^{'}$.*

[1] The conjecture is open even for many subclasses of connected vertex-transitive graphs, including connected Cayley graphs. Moreover, there are only five known examples of vertex- transitive graphs that have a Hamilton path but do not have a Hamilton cycle (K_2, the Peterson graph, the Coxeter graph, and two graphs obtained from the Petersen and Coxeter graphs by replacing each vertex with a triangle).

[2] the poset obtained by ordering all permutations from S_n by containment of their inversion sets, i.e., $\pi < \rho$ for any two permutations π, ρ in the weak order if and only if $inv(\pi) \subseteq inv(\rho)$.

Definition 3 (Lattice quotient) *For a congruence $\equiv$ on a lattice P, the lattice quotient $P/\equiv$ is obtained by taking the equivalence classes as elements, and ordering them by $X < Y$ if and only if there is a representative $\pi \in X$ and a representative $\rho \in Y$ such that $\pi < \rho$ in P.*

Theorem 4 (Hoang, Mütze [13]) *For every lattice congruence $\equiv$ of the weak order on S_n there is a Hamiltonian path in the cover graph of the lattice quotient $S_n/\equiv$.*

Theorem 5 (Pilaud, Santos [24]) *For every lattice congruence $\equiv$ of the weak order on S_n there exists a polytope called the quotientope for $\equiv$, whose graph is exactly the cover graph of the lattice quotient $S_n/\equiv$.*

Remark The polytopes as per Theorem 5 generalize many known polytopes, such as hypercubes, associahedra, permutahedra etc. Thus Theorem 4 and Theorem 5 are examples of "unification" where some Gray code problem for different combinatorial classes can be unified under some general poset.

Going forward we will focus on the case of enumerating all binary strings of a fixed length so that consecutive strings differ in a single coordinate.

2 Binary Gray codes

Definition 6 (Gray code) *A* (cyclic) binary Gray code *of dimension n is a Hamiltonian path (cycle) in Q_n.*

Let us mention some references where Gray codes appeared, mainly from the history of communication:

1872 L. A. Gross described [10] the mathematical background of so called *Chinese ring puzzle* (also known as *Baguenaudier*) which anticipated Gray codes,
1878 E. Baudot[3] used Gray codes to design a telegraphic alphabet (he used a code of dimension 5 to design an alphabet of 31 symbols),
1943 The idea of Gray codes is inherently present in the patent of G. Stiblitz,
1953 F. Gray from Bell Labs patented [6] the code (the patent application was for a pulse code modulation tube which was used for analog transmission of digital signals).

There are also a lot of applications of binary Gray codes such as signal processing, binary counters, data compression, logical circuits, rotary encoders, image processing, campanology etc.

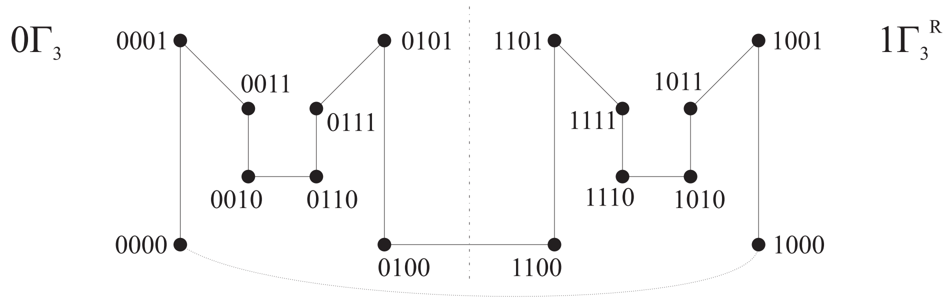
[3]In his honor the unit for communication speed was named *baud*, denoted by Bd. (The number of bauds signifies the number of changes made to a transmission medium per second.)

3 Reflected Gray code

Definition 7 (Binary reflected Gray code (BRGC)) *A binary* reflected Gray code Γ_n *is defined recursively:*

- $\Gamma_0 = \lambda$ *(that is, an empty word),*
- $\Gamma_{n+1} = (0\Gamma_n, 1\Gamma_n^R)$

where Γ_n^R *denotes the reversed sequence* Γ_n*, and* $\alpha\Gamma$ *denotes the sequence obtained from* Γ *by adding the symbol* α *in front of each element of* Γ*.*



We will use the following definitions. Let $\Gamma_n = (v_0, \ldots, v_{2^n-1})$ be the reflected Gray code of dimension n. For $0 \le i < 2^n$ let $(i)_2$ denote the binary representation of i in n bits, and let $\mathtt{sh}_0(x) = 0x_1 \ldots x_{n-1}$ for $x \in \mathbb{Z}_2^n$ (that is, the right shift).

It is not hard to deduce formulae that tell how to compute the i-th element v_i of the reflected Gray code, and vice versa, how to compute the index i of a given element v_i. The proof is left as an exercise. Thus the observation below gives us a ranking/un-ranking algorithm for the reflected Gray code.

Observation 8 *In the reflected Gray code* Γ_n*, for every* $0 \le i < 2^n$

$$v_i = (i)_2 \oplus (\lfloor i/2 \rfloor)_2,$$

$$(i)_2 = v_i \oplus sh_0(v_i) \oplus sh_0^2(v_i) \oplus \ldots \oplus sh_0^{n-1}(v_i).$$

There also exists a simple successor function (giving the next element in the enumeration) for the reflected Gray code, defined as

$$suc(x) = \begin{cases} x \oplus e_1 & \text{if } |x| \text{ is even,} \\ x \oplus e_{i+1} & \text{otherwise} \end{cases} \tag{1}$$

where $x = x_n \cdots x_1$ and $i \ge 1$ is the smallest integer such that $x_i = 1$ ($i := n-1$ for $x = 10\ldots0$). This gives a loopless algorithm for the reflected Gray code. We mention some interesting properties of the reflected Gray code:

– The restriction to any level[4] gives a Gray code for (n, k) combinations, where the "small change" is a swap of a single element,

[4] where levels are based on Hamming weight.

- It follows so called *genlex ordering*[5], i.e., all (bit) strings with the same prefix occur consecutively in the code,
- If $d_H(u,v) = t$, then $\lceil \frac{2^t}{3} \rceil \leq d_{\Gamma_n}(u,v) \leq 2^n - \lceil \frac{2^t}{3} \rceil$, where $d_H(u,v)$ and $d_{\Gamma_n}(u,v)$ stand for the Hamming distance between u and v and the distance between u and v in Γ_n respectively,
- The transition counts for the reflected Gray code are: $2, 2, 4, 8, \cdots, 2^{n-1}$,
- The transition graph corresponding to the reflected Gray code (in Q_n) is a star $K_{1,n-1}$.

4 Generation via permutation languages

In this section, we look at a general framework and how it can be used to generate combinatorial Gray codes. The idea of the framework is to encode objects of the given combinatorial class by permutations with some forbidden patterns.

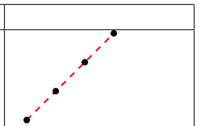
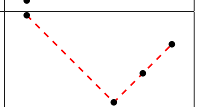
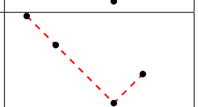
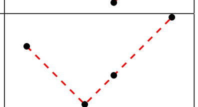
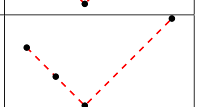
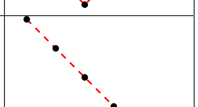
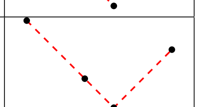
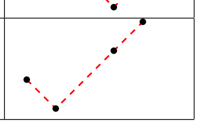
$x_1x_2x_3$	$f(x_1x_2x_3)$	
000	1 2 3 4	
001	4 1 2 3	
011	4 3 1 2	
010	3 1 2 4	
110	3 2 1 4	
111	4 3 2 1	
101	4 2 1 3	
100	2 1 3 4	

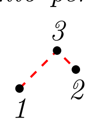
Table 1: Encoding of Γ_3 by (2).

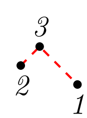
For example consider the encoding of binary strings $f^n : \{0,1\}^n \to S_{n+1}$, given by $f^0(\epsilon) := 1$ (for ϵ denoting the empty string) and f^n for $n \geq 1$ is defined recursively as

$$f^n(x_1x_2\cdots x_n) = \begin{cases} f^{n-1}(x_1x_2\cdots x_{n-1}) \circ (n+1) & \text{if } x_n = 0 \\ (n+1) \circ f^{n-1}(x_1x_2\cdots x_{n-1}) & \text{if } x_n = 1 \end{cases} \tag{2}$$

For example see Table 1 for encoding of bitstrings of length 3.

Definition 9 *A permutation $\pi \in S_n$ is said to contain a pattern $\tau \in S_k$ if $\pi = a_1a_2\cdots a_n$ contains a sub-sequence $a_{i_1}a_{i_2}\cdots a_{i_k}$ that is order-isomorphic[6] to τ.*

Observation 10 *f encodes all binary strings (bijectively) into permutations that avoid peaks, i.e. patterns :* 

and 

(for example see Table 1).

[5]ordering of a set of strings in which strings sharing a common prefix occur together. An example of genlex ordering is the lexicographic ordering.

[6]For example the permutation 5 <u>2 4 3</u> 1 contains the pattern 1 3 2.

Next we describe an algorithm that produces all permutations $L_n \subseteq S_n$ which avoid some given "forbidden" patterns.

Definition 11 *Given a permutation $\pi = a_1 \cdots a_n$ with a substring $a_i \cdots a_j$ with $a_i > a_{i+1}, \cdots, a_j$, a right jump of the value a_i by $j-i$ steps is a cyclic left rotation of this substring by one position to $a_{i+1} \cdots a_j a_i$. Similarly, given a substring $a_i \cdots a_j$ with $a_j > a_i, \cdots, a_{j-1}$, a left jump of the value a_j by $j-i$ steps is a cyclic right rotation of this substring to $a_j a_i \cdots a_{j-1}$.*

Algorithm J (Greedy minimal jumps).

This algorithm attempts to greedily generate a set of permutations $L_n \subseteq S_n$ using minimal jumps starting from an initial permutation $\pi_0 \in L_n$.

J1. [Initialize] Visit the initial permutation π_0.

J2. [Jump] Generate an unvisited permutation from L_n by performing a minimal jump of the largest possible value in the most recently visited permutation. If no such jump exists, or the jump direction is ambiguous, then terminate. Otherwise, visit this permutation and repeat J2.

Theorem 12 ([11]) *If F is a propositional formula of ANDs $\wedge$, ORs $\vee$ and patterns that do not have the largest value at the leftmost or the rightmost position, then the set $S_n(F)$ (set of permutations from S_n that avoid patterns given by formula F) can be generated by Algorithm **J**.*

For example, for $F = 1\ 3\ 2\ \wedge\ 2\ 3\ 1$ (i.e., the peak avoiding permutations) $S_n(F)$ can be generated by Algorithm **J**.

What is interesting is that $f^{-1}(\mathbf{J}(L_n))$, where $\mathbf{J}(L_n)$ is the order in which peak avoiding permutations L_n are generated by Algorithm **J** from the identity permutation $\pi_0 := id$ gives us BRGC. This gives us a simple greedy algorithm for generating BRGC.

J1. Start with $0 \cdots 0$.

J2. Flip the rightmost bit that yields a previously unvisited string and repeat J2.

Following are some other remarks about Algorithm **J**.

- When $F = \emptyset$ (i.e., all permutations), Algorithm **J** yields an ordering of permutations by adjacent transpositions, which coincides with the well-known Steinhaus-Johnson-Trotter order.

- Permutations not containing the pattern $F = 231$ encode Catalan families (eg., binary trees by rotations, triangulations by edge flips, etc.). Algorithm **J** produces some known Gray codes for these classes (for examples see [19]).

– Algorithm **J** works for a much larger family of permutation languages (called *zigzag languages*; see [11]) initialized with the identity permutation.

– Algorithm **J** is not efficient but applying f^{-1} (where f is the encoding function for the combinatorial class) usually leads to a simple greedy algorithm.

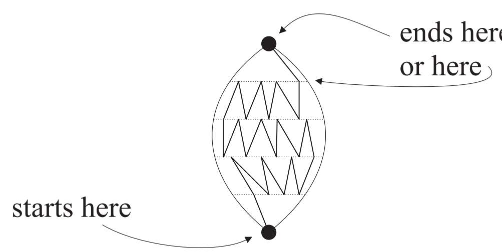
– It can be described where it produces a cyclic order.

– The choice of the initial permutation in the algorithm matters.

5 Monotone Gray codes

Definition 13 (monotone Gray code) *A binary Gray code* $\Gamma = (v_1, \ldots, v_{2^n})$ *is* monotone *if* $|v_i| \leq |v_{i+2}|$ *for every* $1 \leq i \leq 2^n - 2$.

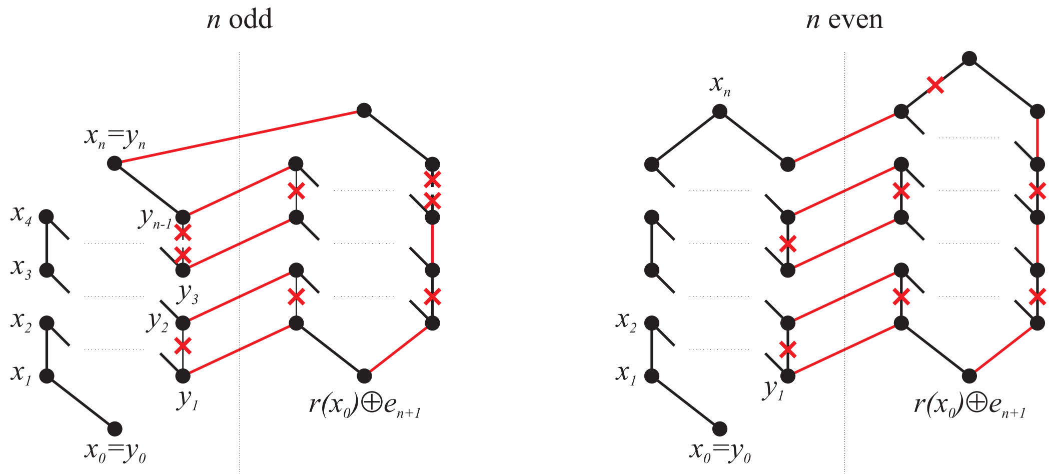
The idea of monotone Gray codes is illustrated in the figure below. The vertices of the hypercube are depicted in the levels by their number of 1's. A monotone Gray starts in **0** or in the first level (but in that case it goes to **0** in the first step), afterwards it always has to oscillate between two consecutive levels until the lower one is entirely covered, and then it continues one level higher. Finally, it ends in **1** or in the penultimate level.



Theorem 14 (Savage, Winkler [27]) Q_n *has a monotone Gray code for every* $n \geq 1$.

Proof We construct recursively by n a monotone Gray code with chains $X = (x_0 = \mathbf{0}, x_1, \ldots, x_n = \mathbf{1})$, $Y = (y_0 = \mathbf{0}, y_1, \ldots, y_n = \mathbf{1})$ such that for every odd i, the edges $x_i x_{i+1}$ and $y_i y_{i+1}$ are the first and the last edge, respectively, of the code between levels i and $i+1$. For $n = 1$ such code clearly exists.

For the induction step, let us assume that we have such a monotone Gray code of dimension n in Q^0_{n+1}. From the previous lectures, we know that there is a rotation $r \in R_n$ such that $r(x_i) = y_i$. In Q^1_{n+1} we take the same code rotated by r. The desired code will be created by joining this two codes together as depicted in the following figure.



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Remark Monotone Gray codes have the maximal number of peaks amongst all Gray codes. A *peak* of a path in Q_n is a subpath (x, y, z) with $|x| = |z|$.

Definition 15 (Order preserving Gray code [14]) *A binary Gray code* $(v_1, v_2, \cdots, v_{2^n})$ *is order preserving if for every* $X \subseteq Y$*,* X *precedes* Y *or* X *follows immediately after* Y *(where* X *and* Y *are vertices of* Q_n *interpreted as sets).*

Remark An order-preserving Gray code is monotone, but the converse is not true.

Problem 16 *Does* Q_n *have an order preserving Gray code for every* $n \geq 2$*?*

Remark It is known that all subsets of $[n]$ of size at most 3 (and some of 4) can be ordered in the way as prescribed by Definition 15, as part of some Gray code.

6 Middle and Central level problems

Definition 17 *The middle level graph in* Q_{2k+1} *is the subgraph induced by levels* k *and* $k+1$*.*

Havel [12] and Buck and Wiedemann [4] asked: Is the middle level graph in Q_{2k+1} Hamiltonian ? The following theorem answers the question positively.

Theorem 18 (Mütze [21]) *The middle level graph in* Q_{2k+1} *is Hamiltonian for every* $k \geq 1$[7].

Theorem 18 is extended by the following two theorems:

[7] For previous progress see [28] and [15], for a shorter proof see [9].

Theorem 19 (Merino, Mička and Mütze [20]) *The middle level graph in Q_{2k+1} has a Hamiltonian cycle that is invariant to cyclic shifts*[8].

Theorem 20 *The middle level graph in Q_{2k+1} is Hamiltonian laceable; that is, for any two vertices x and y from levels k and $k+1$, there is a Hamiltonian path between x and y.*

For $0 \leq k \leq l \leq n$ let $Q_n^{[k,l]}$ denote the subgraph of Q_n induced by levels $k, k+1, \cdots, l$. Then the following theorem generalizes Theorem 18.

Theorem 21 (Gregor, Mička and Mütze [8]) $Q_{2k+1}^{[k-c,k+1+c]}$ *(called the central level graph) is Hamiltonian for every $k \geq 1$ and $0 \leq c \leq k$.*

Definition 22 *A saturating cycle in a bipartite graph is a cycle that visits all vertices in the smaller partition class.*

Definition 23 *A tight enumeration of a (bipartite) subgraph of the cube is a cyclic listing of all its vertices, where the total number of bits flipped is exactly the number of vertices plus the difference in size between the two partition classes.*

Corollary 24 $Q_n^{[k,l]}$ *has a saturating cycle and a tight enumeration for any $0 \leq k \leq l$.*

7 Symmetric chain decomposition (SCD)

In this section, we switch to the language of Boolean lattices $\mathcal{B}_n = (\mathcal{P}([n]), \subseteq)$.

Definition 25 *A* symmetric chain *in $\mathcal{B}_n$ is a sequence of sets $X_k \subset X_{k+1} \subset \ldots \subset X_{n-k}$ with $|X_i| = i$ for every $k \leq i \leq n-k$.*

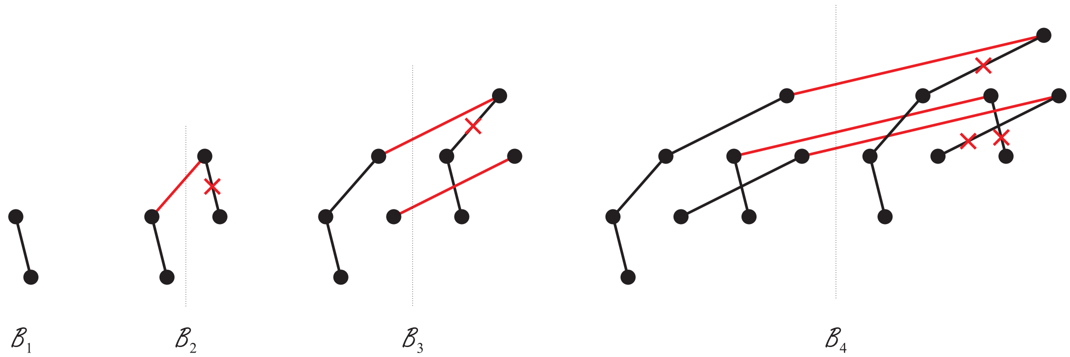
Remark Equivalently, a symmetric chain in Q_n is a path $X_k, X_{k+1}, \cdots, X_{n-k}$ such that $|X_i| = i$ for every $k \leq i \leq n-k$.

Proposition 26 *$\mathcal{B}_n$ has a decomposition into symmetric chains for every $n \geq 1$.*

Proof We give the proof by describing the symmetric decomposition due to Greene and Kleitman[7] (called the Greene-Kleitman SCD). We proceed by induction on n. For $n = 1$ the statement clearly holds since B_1 consists of a single symmetric chain $(\emptyset, \{1\})$.

For a symmetric chain $C = (X_k, \ldots, X_{n-k})$ in $\mathcal{B}_n$ let $C' = (X_k, \ldots, X_{n-k}, X_{n-k} \cup \{n+1\})$ and if $k < n/2$, let $C'' = (X_k \cup \{n+1\}, \ldots, X_{n-k-1} \cup \{n+1\})$. If $k = n/2$ then C'' is undefined. Note that C' and C'' are symmetric chains in B_{n+1}, see the figure below. Observe that if $\mathcal{C}$ is a symmetric chain decomposition of $\mathcal{B}_n$, then $\{C', C'' \mid C \in \mathcal{C}\}$ is a symmetric chain decomposition of $\mathcal{B}_{n+1}$.

[8]The problem was proposed by Knuth [17].



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By definition, every symmetric chain intersects the middle level (both middle levels if n is odd). Thus, there is exactly $\binom{n}{\lfloor n/2 \rfloor}$ chains in a symmetric chain decomposition. On the other hand, every pair of sets from the same level is incomparable (by inclusion); that is, a level forms an antichain. Since every antichain contains at most one set from each chain of decomposition, we obtain the classical Sperner theorem.

Corollary 27 (Sperner [31]) *The maximal size of an antichain in $\mathcal{B}_n$ is $\binom{n}{\lfloor n/2 \rfloor}$.*

Theorem 28 (Gregor, Mička and Mütze [8]) *For every $n \geq 2$ the Greene-Kleitman SCD extends to a Hamiltonian cycle in Q_n.*

Remark Gray codes corresponding to Hamiltonian cycles obtained by extending symmetric chain decompositions have a minimal number of peaks.

Conjecture 29 ([8]) *Every SCD can be extended to a Hamilton cycle in Q_n.*

8 Transitional Sequences

Definition 30 (Transitional sequence) *A* transitional sequence *of a cyclic Gray code $\Gamma = (v_1, \dots, v_{2^n})$ is a sequence $t(\Gamma) = (t_1, \dots, t_{2^n})$ over $[n]$ such that $v_i \oplus v_{i+1} = e_{t_i}$ for every $1 \leq i \leq 2^n$ (we consider $v_{2^n+1} = v_1$).*

Definition 31 (Transition counts) *For a cyclic Gray code $\Gamma = (v_1, \dots, v_{2^n})$, let c_i ($i \in [n]$) denote the number of occurrences of i in $t(\Gamma)$. Then $c_1, c_2, \cdots, c_n$ are called the transition counts for Γ.*

Definition 32 (Balanced Gray code) *A cyclic Gray code Γ is called* (almost) balanced *if $c_i = \frac{2^n}{n}$ for every $i \in [n]$ (respectively, $\left|c_i - \frac{2^n}{n}\right| \leq 2$) where $c_1, c_2, \cdots, c_n$ are its transition counts.*

Theorem 33 ([1, 25, 32]) *For every $n \geq 1$ there is an almost balanced Gray code. In particular, for $n = 2^k$ there is a balanced Gray code.*

Theorem 34 (Perezhogin [23]) *Let $c_1 \le c_2 \le \cdots \le c_n$ be the supposed transition counts. Q_n has a Hamiltonian cycle with these counts if and only if* [9]

$$\begin{aligned} &\sum_{i=1}^{n} c_i = 2^n \quad \text{and} \\ &\sum_{i=1}^{k} c_i \ge 2^k \quad \text{for } 1 \le k \le n\,. \end{aligned} \tag{3}$$

The necessity part of the above theorem can be checked by considering a projection of a such code into subcubes of dimension k.

Definition 35 (Transition graph) *The* graph $G(\Gamma) = (V, E)$ induced by *a Gray code Γ has the vertex set $V = [n]$ and $ij \in E$ if i and j are consecutive in the transitional sequence $t(\Gamma)$.*

It is known [5] that for every $n \le 6$ there is a Gray code inducing the path P_n but there is no Gray code inducing P_7. Despite considerable effort, no Gray codes inducing longer paths have been found. That leads to the following conjecture.

Conjecture 36 (Bultena, Ruskey [5], Slater [29]) *For every $n \ge 7$, the path P_n is not inducible by a Gray code in Q_n.*

Another interesting question [5] is whether there is a Gray code that induces a cycle C_n for every (large enough) n.

Definition 37 *For a Gray code Γ, the* run length $r(\Gamma)$ *is the minimal number of steps in which some direction repeats in $t(\Gamma)$. Furthermore, a* maximal run length *in Q_n is*

$$r(n) = max\{r(\Gamma) \mid \Gamma \text{ is a Gray code in } Q_n\}.$$

Problem 38 ([17]) *Determine $r(n)$ for every n.*

The values of $r(n)$ are known for $n \le 7$, for example $r(5) = 4$. Furthermore, asymptotically it holds $r(n) = n - O(\log n)$.

9 Antipodal Gray codes

Definition 39 (Antipodal Gray code) *A Gray code Γ in Q_n is* antipodal *if every pair of antipodal vertices in Q_n is at distance exactly n in Γ.*

Theorem 40 (Kilian, Savage [16]) *An antipodal Gray code in Q_n exists if $n = 2^k$ and only if n is even.*

Let us note that for $n = 2^k$ the proof is constructive. On the other hand, it is known [16] that there is no antipodal Gray code for $n = 6$. Hence, it would be interesting to know whether $n = 2^k$ is also a necessary condition.

Problem 41 ([16]) *Does an antipodal code in Q_n exist only if $n = 2^k$?*

[9] The necessity part can be checked by considering a projection of such code into a k-dimensional subcube.

Notes

For Gray codes in a broader context, we refer to an excellent survey by Mütze [22] that updates the previous survey of Savage [26]. A part of Knuth's famous monograph [17] is devoted to Gray codes which is recommended for further study. For example, details of the proof of Observation 8 can be found there. Proposition 26 is involved in Kleitman's solution of the Littlewood-Offord problem, for details see e.g. [3].

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