Hypercube problems

Lecture 11

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1 Independent spanning trees

Spanning trees T_1, T_2 of a graph G rooted at the same vertex r are said to be (vertex) *independent* if for every $v \in V(G)$ the paths from v to the root r in T_1, T_2 are inner vertex-disjoint.

This definition is motivated by the problem of communication in a network with a source vertex (the root) that often needs to spread information to target vertices in pieces through disjoint routes. Ideally, if there are many pairwise independent spanning trees rooted at the source vertex, we can use them as fixed communication networks for this task regardless of the target vertices.

Clearly, a vertex-transitive graph G cannot have more than $\kappa(G)$ (vertex-connectivity) pairwise independent spanning trees (shortly *ISTs*) rooted at the same vertex. We will see that in the hypercube, we can construct up to $n = \kappa(Q_n)$ ISTs from binomial trees; see Figure 1 for example in Q_3. First, we need some definitions to describe binomial trees in hypercubes.

For $0 \le i \le n$ let $[n]^i$ denote the set of all sequences of length i over $[n]$ without repetition. For $S = (s_1, \ldots, s_i) \in [n]^i$ let $S' = \{s_1, \ldots, s_i\}$ and $sh_{j-1}(S) = (s_j, \ldots, s_i, s_1, \ldots, s_{j-1})$ for $1 \le j \le i$; that is, S' denotes the set of elements from the sequence S and $sh_{j-1}(S)$ denotes left rotation $j-1$ times.

Definition 1 *For a vertex $r \in V(Q_n)$ and a sequence $S \in [n]^i$ the* binomial S-tree $B_i(r, S)$ *of order i rooted at r is defined as follows.*

- *$B_0(r, \emptyset)$ consists of the single vertex r,*
- *$B_{i+1}(r, S)$ is obtained by joining roots of $B_i(r, T)$ and $B_i(r \oplus e_j, T)$ where $S = (T, j)$.*

For example see $B_3(e_1, (1, 2, 3))$ on Figure 1. Clearly, $B_n(r, S)$ is a spanning tree of Q_n for any permutation S of $[n]$.

Note that in the following observation a subsequence is not meant to be necessarily contiguous.

Observation 2 *The transitional sequence of every path to r in $B_i(r, S)$ is a subsequence of S.*

The desired n ISTs in Q_n are obtained by changing the root in following binomial trees.

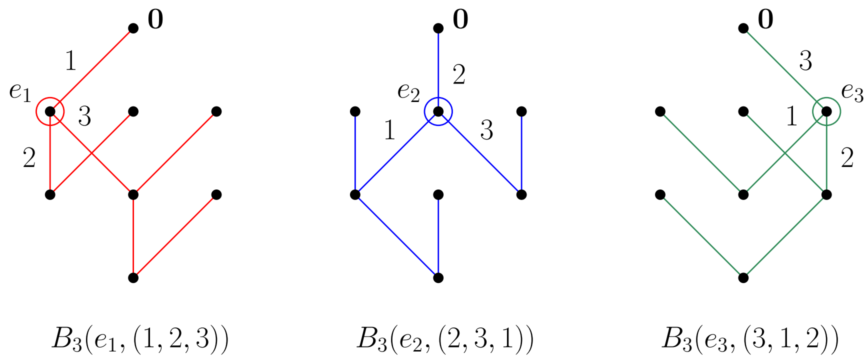

Figure 1: Three pairwise independent spanning trees of Q_3 rooted at the vertex $\mathbf{0}$.

Theorem 3 *For* $1 \leq i \leq n$ *let* $T_i = B_n(e_i, sh_{i-1}(S))$ *where* $S = (1, 2, \ldots, n)$. *Then* $T_1, \ldots, T_n$ *rooted at the vertex* $\mathbf{0}$ *are pairwise independent spanning trees.*

Proof First note that every (nontrivial) path in T_i to the vertex $\mathbf{0}$ ends with the edge $e_i\mathbf{0}$. Suppose that paths from a vertex u to $\mathbf{0}$ in T_i, T_j intersect in an inner vertex v for some distinct $u, v \in V(Q_n)$ and $i, j \in [n]$. Let A_i and B_i be the transitional sequences of the paths in T_i from u to v and from v to $\mathbf{0}$, respectively. Similarly, we define A_j and B_j for the paths in T_j. Clearly, $A'_i = A'_j \neq \emptyset$ and $B'_i = B'_j \neq \emptyset$ as any $k \in [n]$ repeats in none of A_i, B_i, A_j, B_j.

Assume that $i < j$ and let $C = (i, \ldots, j-1)$ and $D = (j, \ldots, n, 1, \ldots, i-1)$. Thus, $sh_{i-1}(S) = (C, D)$ and $sh_{j-1}(S) = (D, C)$. By Observation 2, (A_i, B_i) is a subsequence of (C, D, i) and (A_j, B_j) is a subsequence of (D, C, j). Since B_i ends with i, it follows that i appears in B_j, and consequently, A_j is a subsequence of D. Similarly, since B_j ends with j, it follows that j appears in B_i, and consequently, A_i is a subsequence of C. This contradicts $A'_i = A'_j \neq \emptyset$ and $C' \cap D' = \emptyset$. ■

Remark The path from a vertex v to $\mathbf{0}$ in T_i has the (shortest) length $|v| = d_H(v, \mathbf{0})$ if $v_i = 1$, and $|v| + 2$ if $v_i = 0$. (In the latter case, it starts and ends with the direction i.) Hence, for every vertex v the inner vertex-disjoint paths from v to $\mathbf{0}$ in the above ISTs have optimal total length.

By vertex-transitivity of the hypercube we can choose any root for ISTs.

Corollary 4 *For every* $r \in V(Q_n)$ *there are* n *pairwise independent spanning trees in* Q_n *rooted at* r *with optimal total length of paths from any vertex to the root* r.

Remark It is worth mentioning that binomial trees among all spanning trees of hypercubes have smallest average message delay in the following sense. The *total distance* of a graph G (also known as *Wiener index*) is $td(G) = \sum_{\{u,v\}} d(u, v)$. It is known that B_n has minimal total distance over all spanning trees of Q_n [1]. In particular,

$$td(B_n) = 2td(Q_n) - \binom{2^n}{2}. \tag{1}$$

The proof of (1) is left as an exercise.

2 Matrix-tree theorem

The following classic result in spectral graph theory gives us the exact number of spanning trees in a graph from the spectrum of its Laplacian matrix. The *Laplacian matrix* of an undirected simple graph G with $V(G) = \{v_1, \dots, v_m\}$ is the $m \times m$ matrix $L(G) = (l_{ij})$

$$l_{ij} = \begin{cases} deg_G(v_i) & \text{if } i = j, \\ -1 & \text{if } v_i v_j \in E(G), \\ 0 & \text{otherwise.} \end{cases}$$

Note that if G is n-regular, then $L(G) = nI - A(G)$ where I is the identity matrix and $A(G)$ is the adjacency matrix of G. Thus, if $0 = \lambda_1 \leq \cdots \leq \lambda_p$ are the eigenvalues of $L(G)$, then $n = n - \lambda_1 \geq \cdots \geq n - \lambda_p$ are the eigenvalues of $A(G)$.

Theorem 5 (Matrix-Tree) *An undirected simple graph G has exactly*

$$\frac{1}{|V(G)|} \lambda_2 \cdots \lambda_p$$

spanning trees where $0 = \lambda_1 \leq \cdots \leq \lambda_p$ are the eigenvalues of $L(G)$.

For a proof we refer to the classical monograph of Stanley [6].

3 Spectrum

We will determine the spectrum of the hypercube by guessing and verifying the eigenvectors of its Laplacian matrix. The eigenvectors correspond to so called character functions that are used as a Fourier basis in harmonic analysis of boolean functions.

For convenience we switch to $\langle \{-1, 1\}, \cdot, id, 1 \rangle \simeq \langle \{0, 1\}, +, id, 0 \rangle$. Let Ω_n be the vector space of all functions $f : \{-1, 1\}^n \to \mathbb{R}$. Each $f \in \Omega_n$ can be seen as a vector in $\mathbb{R}^{2^n}$. The standard basis of Ω_n is $\{g_x\}_{x \in \{-1,1\}^n}$ where $g_x \in \Omega_n$ such that

$$g_x(y) = \begin{cases} 1 & \text{if } y = x, \\ 0 & \text{if } y \neq x, \end{cases}$$

for every $y \in \{-1, 1\}^n$. We consider a linear transformation $\Phi : \Omega_n \to \Omega_n$ given by

$$(\Phi f)(x) = n f(x) - \sum_{i \in [n]} f(x \oplus e_i)$$

where $e_i \in \{-1, 1\}^n$ has -1 exactly on the ith coordinate and $x \oplus y = (x_1 y_1, \dots, x_n y_n)$.[1]

Observation 6 *The matrix of the linear transformation Φ is exactly the Laplacian matrix $L(Q_n)$ (with respect to the same ordering of vertices of Q_n as the ordering of the basis $\{g_x\}$).*

[1] Then e_i and $\oplus$ have the same meaning as we are used to.

For a set $S \subseteq [n]$ we define $\chi_S : \{-1,1\}^n \to \{-1,1\}$, called *a character at* S, by

$$\chi_S(x) = \prod_{i \in S} x_i \quad \text{if } S \neq \emptyset, \ \text{ and } \ \chi_\emptyset(x) = 1$$

for every $x \in \{-1,1\}^n$. Now we verify that χ_S are eigenvectors of $L(Q_n)$, as

$$\begin{aligned}(\Phi\chi_S)(x) &= n\chi_S(x) - \sum_{i\in[n]} \chi_S(x \oplus e_i) \\ &= (n - ((n-|S|) - |S|)\chi_S(x) \quad \text{since } \chi_S(x \oplus e_i) \begin{cases} -\chi_S(x) & \text{if } i \in S \\ \chi_S(x) & \text{if } i \notin S \end{cases} \\ &= 2|S|\chi_S(x).\end{aligned}$$

It can be shown that all characters are linearly independent, hence they are all eigenvectors of $L(Q_n)$. Actually, they form an orthogonal basis of Ω_n, which is left as an exercise.

Thus, the corresponding eigenvalues of $L(Q_n)$ are $\{0, 2, \ldots, 2n\}$, the eigenvalue $2i$ with multiplicity $\binom{n}{i}$. Applying the matrix-tree theorem, we obtain the exact number of spanning trees of Q_n.

Corollary 7 *For every* $n \geq 1$ *the number of spanning trees of* Q_n *is*

$$\frac{1}{2^n} \prod_{i\in[n]} (2i)^{\binom{n}{i}}. \tag{2}$$

Problem 1 *Find a direct combinatorial proof of the formula* (2).

4 Embedding binary trees

A complete binary tree on $2^n - 1$ vertices is not almost balanced if $n \geq 3$. Therefore, it does not have an embedding (that is, an injective homomorphism) to Q_n. However, by splitting the root into two adjacent vertices and assigning each subtree to its own root we obtain *two-rooted complete binary tree* on 2^n vertices that is balanced, see Figure 2(a).

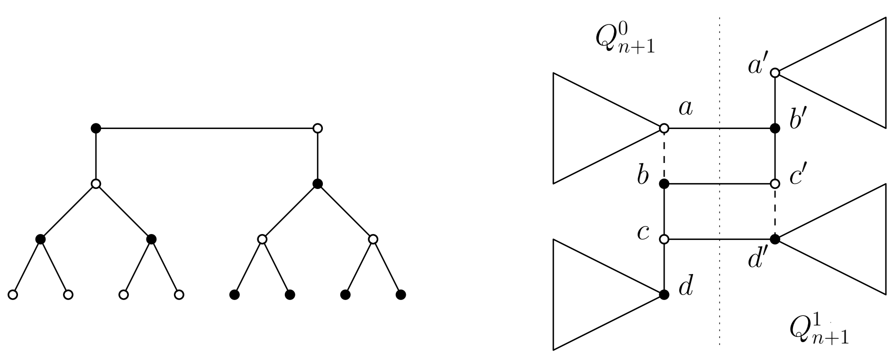

Figure 2: (a) a two-rooted complete binary tree, (b) an embedding by induction.

Theorem 8 (Havel and Liebl [3]) *The two-rooted complete binary tree on 2^n vertices is a spanning tree of Q_n.*

Proof The statement is trivially true for $n = 1, 2$. We construct the embedding of the tree of order $n + 1$ by induction, see Figure 2(b). By 2-arc transitivity of Q_n, we choose embeddings of the trees of order n into Q^0_{n+1} and Q^1_{n+1} so that (a, b, c) is adjacent to (b', c', d'). Then b and c' are the roots of the tree of order $n + 1$ with sons c and b', respectively. ■

A *general embedding* of a graph G to a graph H is a pair of mappings $f : V(G) \to V(H)$ and $g : E(G) \to$ *paths in* G such that $g(uv)$ is a path between $f(u)$ and $f(v)$. This is a widely-used notion important e.g. for study of simulations. The *dilation* of an edge $uv \in E(G)$ is the length of the path $g(uv)$. The dilation of the embedding is the maximal dilation of an edge of G.

Corollary 9 *A complete binary tree on $2^n - 1$ vertices has an injective general embedding to Q_n with dilation 1, except of one edge from the root which has dilation 2.*

There are many results (and open questions) on embeddings into hypercubes. In particular for trees, there are results e.g. for quasistars, caterpillars, ternary trees.

Conjecture 10 (Havel [2]) *Every balanced tree on 2^n vertices with maximal degree at most 3 is a spanning tree of Q_n.*

Notes

The construction from Theorem 3 is deduced from Liu et al. [4]. There are earlier constructions of n ISTS for Q_n [5] but they do not necessarily have an optimal total length as stated by Corollary 4.

For a proof of the matrix-tree theorem and its history we refer to an excellent monograph of Stanley [6] which is recommended for further reading. Section 5 including Problem 1 is mainly taken from Example 5.6.10 in [6], although with different notation.

References

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