Hypercube problems

Lecture 12

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1 Shadows

In the Boolean lattice $\mathcal{B}_n = (\mathcal{P}([n]), \subseteq)$ we denote $X = [n]$ and for $0 \le r \le n$ the r-th *level* is $X^{(r)} = \{A \subseteq X;\ |X| = r\}$. For a set system $\mathcal{A} \subseteq X^{(r)}$, the *lower (upper) shadow* of $\mathcal{A}$ is

$$\partial_l \mathcal{A} = \{B \in X^{(r-1)};\ B \subset A \text{ for some } A \in \mathcal{A}\} \qquad \text{(lower shadow)},$$
$$\partial_u \mathcal{A} = \{B \in X^{(r+1)};\ B \supset A \text{ for some } A \in \mathcal{A}\} \qquad \text{(upper shadow)}.$$

By default, $\partial \mathcal{A}$ denotes the lower shadow. We will determine the minimal $|\partial \mathcal{A}|$ in terms of $|\mathcal{A}|$, n, r. For this end, we first look into properties of the colexicographical[1] order on $X^{(r)}$.

Definition 1 (colexicographical (colex) order) $A <_{colex} B$ *if* $\max(A \triangle B) \in B$.

The colex order can be extended on the whole $\mathbb{N}_+^{(<\omega)}$ (the set of all finite subsets of positive integers) by the map $\varphi(A) = \sum_{a \in A} 2^a$ and using the natural order of $\mathbb{N}$. Observe that the initial segment of the colex order on $X^{(r)}$ is independent[2] on n. For example for $r = 3$ we have:

$$\{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}, \{1,2,5\}, \{1,3,5\}, \{2,3,5\}, \{1,4,5\}, \{2,4,5\}, \{3,4,5\}, \ldots$$

To describe precisely the (lower) shadow of an initial segment of the colex order on $X^{(r)}$ we use the following definition. For $1 \le s \le m_s < m_{s+1} < \cdots < m_r$ let

$$B^{(r)}(m_r, \ldots, m_s) = \bigcup_{j=s}^{r} \Big([m_j]^{(j)} \cdot \{m_{j+1}+1, \ldots, m_r+1\} \Big)$$

where $\mathcal{A} \cdot M = \{A \cup M;\ A \in \mathcal{A}\}$. For example,

$$B^{(3)}(4,2,1) = [4]^{(3)} \cup ([2]^{(2)} \cdot \{5\}) \cup ([1]^{(1)} \cdot \{3,5\}) = [4]^{(3)} \cup \{\{1,2,5\}, \{1,3,5\}\}.$$

[1] Sometimes (e.g in [3]), this order is called *reversed lexicographical (revlex)* since it is a lexicographical order with respect to the reversed (natural) order of $[n]$. Here, the *lexicographical (lex) order* on $X^{(r)}$ is given by $A <_{lex} B$ if $\min(A \triangle B) \in B$; that is, lex is the standard lexicographical order of support vectors. However, note that Bollobás [1] defines the lexicographical order on $X^{(r)}$ reversely, by $A <_{lex'} B$ if $\min(A \triangle B) \in A$; that is, $A <_{lex'} B$ if and only if $B <_{lex} A$.

[2] This is not true neither for the lex nor for the lex' order. This explains why it is preferred to used the colex order instead of lex, although for fixed n reversing the order of $[n]$ maps colex to lex.

Observe that $B^{(r)}(m_r, \ldots, m_s)$ is the initial segment of the colex order of $\mathbb{N}_+^{(r)}$ of length

$$|B^{(r)}(m_r, \ldots, m_s)| = \sum_{j=s}^{r} \binom{m_j}{j}. \tag{1}$$

On the other hand, it can be shown that every initial segment can be written as $B^{(r)}(m_r, \ldots, m_s)$, and that $m_r, \ldots, m_s$ are determined uniquely from the segment length.

The above (rather technical) description is justified by the following observation.

Observation 2 *For every r, $s \le m_s < \cdots < m_r$*

$$\partial B^{(r)}(m_r, \ldots, m_s) = B^{(r-1)}(m_r, \ldots, m_s).$$

In words, the shadow of the initial segment of the colex order of $X^{(r)}$ is the initial segment of the colex order of $X^{(r-1)}$ described by the same parameters.

Why are initial segments of the colex order so important? Because they have smallest shadows among all set systems of $X^{(r)}$ with fixed size. This classical result is well-known as the Kruskal-Katona theorem. To state it properly, we define $\partial^{(r)} : \mathbb{N} \to \mathbb{N}$ by

$$\partial^{(r)}(m) = \sum_{j=s}^{r} \binom{m_j}{j-1}$$

where $1 \le s \le m_s < \cdots < m_r$ are the unique numbers such that $m = \sum_{j=s}^{r} \binom{m_j}{j}$. That is, $\partial^{(r)}(m) = |\partial B^{(r)}(m_r, \ldots, m_s)|$ is the shadow size of the first m r-sets in the colex order.

Theorem 3 (Kruskal-Katona [7, 8]) *For every $\mathcal{A} \subseteq X^r$ where $r \ge 1$,*

$$|\partial \mathcal{A}| \ge \partial^{(r)}(|\mathcal{A}|).$$

Moreover, if $|\mathcal{A}| = \binom{m_r}{r}$ for some $m_r \ge r$ then equality holds if and only if $\mathcal{A} = [m_r]^{(r)}$.

The original proof has been simplified over the years. We refer to Bollobás [1] for a proof based on the compression operator, due to Frankl [4].

For upper shadows we apply Kruskal-Katona theorem on the family of complements. Since complements of a prefix of the colex order on $X^{(n-r)}$ form a suffix of the colex order on $X^{(r)}$, we obtain the following corollary.

Theorem 4 *Let $1 \le r \le n-1$, $\mathcal{A} \subseteq X^{(r)}$, and let $\mathcal{B}$ be the set of the last $|\mathcal{A}|$ elements of $X^{(r)}$ in the colex order. Then*

$$|\partial_u \mathcal{A}| \ge |\partial_u \mathcal{B}|.$$

2 Intersecting families

A family $\mathcal{A} \subseteq \mathcal{P}(X)$ is *intersecting* if $A_1 \cap A_2 \neq \emptyset$ for every $A_1, A_2 \in \mathcal{A}$. We restrict here only to showing that Kruskal-Katona theorem implies Erdős-Ko-Rado theorem, a classical result on the maximal size of an intersecting family in the r-th level.

For $x \in X$ and $1 \le r \le n$ let $X_x^{(r)} = (X \setminus \{x\})^{(r-1)} \cdot \{x\}$; that is, X_x^r consists of all r-sets containing x.

Theorem 5 (Erdős-Ko-Rado [2]) *For every intersecting* $\mathcal{A} \subseteq X^{(r)}$ *where* $2 \le r < n/2$,

$$|\mathcal{A}| \le \binom{n-1}{r-1} \tag{2}$$

with equality if and only if $\mathcal{A} = X_x^{(r)}$ *for some* $x \in X$.

Proof Let $\mathcal{B} = \{\overline{A};\ A \in \mathcal{A}\}$; that is, $\mathcal{B}$ is the family of complements in $X^{(n-r)}$. If $A \in \mathcal{A}$ and $B \in \mathcal{B}$ then $A \nsubseteq B$. Consequently, $\partial^{n-2r}\mathcal{B} \cap \mathcal{A} = \emptyset$ where ∂^{n-2r} denotes the shadow taken $n-2r$ times.

Assume equality in (2). Since $X_x^{(r)}$ is a maximal (with respect to inclusion) intersecting family, it suffices to show that $\mathcal{A} = X_x^{(r)}$. By Theorem 3, $|\partial^{n-2r}\mathcal{B}| \ge \binom{n-1}{r}$ with equality if and only if $B = (X \setminus \{x\})^{(n-r)}$ for some $x \in X$ which is equivalent to $\mathcal{A} = X_x^{(r)}$. ■

3 Isoperimetric inequalities

The notion of isoperimetry ("having the same perimeter") comes from geometry. The isoperimetric inequality in the plane is the relation $4\pi A \le L^2$ between area A of a planar region and length L of its enclosing curve (with equality if and only if the curve is a circle)[3].

In a graph $G = (V, E)$ we consider subsets $S \subseteq V$ of vertices (instead of area in the plane) and we are interested in their *vertex* and *edge boundaries*[4]:

$$\begin{aligned} \partial_v S &= N(S) \setminus S = \{v \in V \setminus S;\ uv \in E \text{ for some } u \in S\} && \text{(vertex boundary)},\\ \partial_e S &= E(S, \overline{S}) = \{uv \in E;\ u \in S, v \notin S\} && \text{(edge boundary)}. \end{aligned}$$

In general, isoperimetric inequality is a lower bound on $|\partial S|$ in terms of $|S|$ and other parameters of G. Ideally, we determine for all $1 \le m < |V|$ the values of

$$\Phi_v^G(m) = \min_{S \subseteq V, |S| = m} |\partial_v S|, \quad \Phi_e^G(m) = \min_{S \subseteq V, |S| = m} |\partial_e S|,$$

called the *vertex* and *edge isoperimetric parameters* of G. (We study these functions only for the hypercube, so we omit the superscript G).

4 Vertex-isoperimetric problem

A *Hamming sphere* with a center $C \subseteq X$ is a family $\mathcal{A} \subseteq \mathcal{P}(X)$ such that

$$\{B \subseteq X;\ d_H(B, C) \le r\} \subseteq \mathcal{A} \subset \{B \subseteq X;\ d_H(B, C) \le r + 1\}$$

for some $0 \le r \le n$. That is, $\mathcal{A}$ contains the Hamming ball of radius r and is strictly contained in the Hamming ball of radius $r+1$, both centered in C.

The (Hamming) distance between two nonempty families $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}(X)$ is $d_H(\mathcal{A}, \mathcal{B}) = \min\{d_H(A, B);\ A \in \mathcal{A}, B \in \mathcal{B}\}$. The following theorem claims that Hamming spheres with antipodal centers are among the "furthest" families of given sizes.

[3]This inequality was known to ancient Greeks, but it was rigorously proved no early than in 19th century.

[4]The notion of shadow is very close to the notion of vertex boundary in the hypercube. We keep the symbol ∂ commonly used for both these notions.

Theorem 6 *For nonempty $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}(X)$ there are Hamming spheres $\mathcal{A}_0$, $\mathcal{B}_0$ centered at $\emptyset$ and X, respectively, with $|\mathcal{A}_0| = |\mathcal{A}|$, $|\mathcal{B}_0| = |\mathcal{B}|$ such that $d_H(\mathcal{A}_0, \mathcal{B}_0) \geq d_H(\mathcal{A}, \mathcal{B})$.*

The proof is based on compressing $\mathcal{A}$ towards $\emptyset$, and simultaneously $\mathcal{B}$ towards X iteratively by some coordinate $x \in X$. For details we refer to [1].

By the above theorem, a solution to the vertex-isoperimetric problem in Q_n is attained by a Hamming sphere. Indeed, for a nonempty $\mathcal{A} \subset \mathcal{P}(X)$ of a fixed size with the smallest vertex-boundary we take

$$\mathcal{B} = \begin{cases} \partial_v \mathcal{A} & \text{if } \mathcal{A} \cup \partial_v \mathcal{A} = \mathcal{P}(X), \\ \mathcal{P}(X) \setminus (\mathcal{A} \cup \partial_v \mathcal{A}) & \text{otherwise.} \end{cases}$$

In the first case, we have $d_H(\mathcal{A}, \mathcal{B}) = 1$ and $\partial_v \mathcal{A}_0 \subseteq \mathcal{B}_0$. In the latter case, we have $d_H(\mathcal{A}, \mathcal{B}) = 2$ and $\partial_v \mathcal{A}_0 \subseteq \mathcal{P}(X) \setminus (\mathcal{A}_0 \cup \mathcal{B}_0)$. Hence, in both cases $\mathcal{A}_0$ does not have a larger vertex-boundary than $\mathcal{A}$.

Kruskal-Katona theorem for upper shadows tells us which sets from the last level $X^{(r+1)}$ we should take in a Hamming sphere $\mathcal{A}$ to minimize its vertex boundary, see the figure below.

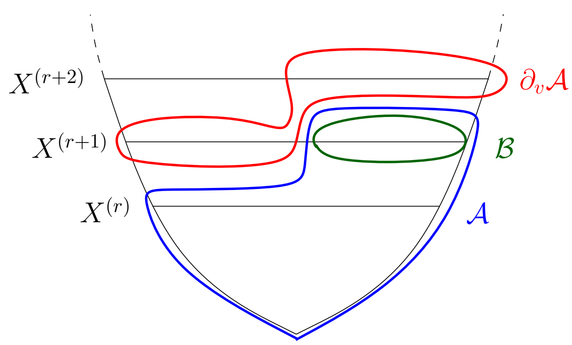

Figure 1: $\mathcal{B} = \mathcal{A} \cap X^{(r+1)}$ is the final segment of $X^{(r+1)}$ in the colex order.

To summarize the solution of the vertex isoperimetric problem, we define the following order on $\mathcal{P}(X)$.

Definition 7 (simplicial order) *$A < B$ if $|A| < |B|$ or $|A| = |B|$ and $\max(A \triangle B) \in A$.*

Theorem 8 (Harper [6]) *For every $\mathcal{A} \subseteq \mathcal{P}(X)$ it holds $|\partial_v \mathcal{A}| \geq |\partial_v \mathcal{S}|$ where $\mathcal{S}$ is the system of first $|\mathcal{A}|$ sets in the simplicial order. In particular, if $\mathcal{A} = \sum_{i=0}^{r} \binom{n}{i}$ for some r, then $\partial \mathcal{A} \geq \binom{n}{r+1}$.*

Remark Since $\mathcal{S} \cup \partial \mathcal{S}$ is a prefix of the simplicial order if S is a prefix of the simplicial order, the above theorem holds also for the vertex boundary in a general distance d.

Notes

The above classical results can be found in almost any book on combinatorics of finite sets. We refer to the presentation given by Bollobás [1]. Theorem 6 is due to Frankl and Füredi [5] who simplified Harper's proof of Theorem 8.

References

[1] B. Bollobás, *Combinatorics: set systems, hypergraphs, families of vectors, and combinatorial probability*, Cambridge University Press, Cambridge, 1986.

[2] P. Erdős, C. Ko., and R. Rado, *Intersection theorems for systems of finite sets*, Quart. J. Math. Oxford 12 (1961), 313–320.

[3] S. Felsner and M. Massow, *Linear extension diameter of downset lattices of two-dimensional posets*, SIAM J. Discrete Math. 25 (2011), 112-129.

[4] P. Frankl, *A new short proof of the Kruskal-Katona theorem*, Discrete Math. 48 (1984), 327–329.

[5] P. Frankl and Z. Füredi, *A short proof of a theorem of Harper about Hamming spheres*, Discrete Math. 34 (1981), 311–313.

[6] L. H. Harper, *Optimal numberings and isoperimetric problems on graphs*, J. Combin. Theory 1 (1966), 385–394.

[7] G. O. H. Katona, *A theorem on finite sets*, in Theory of Graphs (P. Erdős and G. O. H. Katona, eds.), Akadémiai Kiadó, Budapest, 1968, 187–207.

[8] J. B. Kruskal, *The number of simplices in a complex*, in Mathematical Optimization Techniques, Univ. California Press, Berkeley, 1963, 251–278.