Hypercube problems

Lecture 13

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We continue in study of isoperimetric parameters of hypercubes started in the previous lecture. Recall that it remains to determine the *edge isometric parameter* $\Phi_e(m)$ which gives the minimal size of an *edge boundary* $E(S,\overline{S})$ over all sets S of m vertices in Q_n.

1 Edge-isoperimetric problem

In hypercubes we may equivalently consider a dual problem of determining

$$f_n(m) = \max\{|E(Q_n[S])| : \; S \subseteq V(Q_n), |S| = m\},$$

the maximal number of edges in Q_n spanned by m vertices. Indeed, since Q_n is n-regular,

$$\Phi_e(m) = nm - 2f_n(m).$$

Intuitively, one would guess that for $m = 2^k$ the maximal number of edges is attained by a subcube of dimension k. We will see that this intuition is correct. Let us start by a definition of an auxiliary function.

Definition 1 *For $0 \le l < m$ let*

$$f(l,m) = \sum_{i=l}^{m-1} h(i)$$

where $h(i)$ is the number of 1's in the binary representation of i.

A proof of the following lemma is left as an exercise. Details can be found in [2].

Lemma 2 *If $1 \le k \le l$ then $f(l, l+k) \ge f(0,k) + k$.*

At first it may not be clear what this seeming technical definition has to do with our problem. But it turns out that it describes optimal solution given by prefixes of the lexicographical ordering. For $X = [n]$ the *lexicographical order* on $\mathcal{P}(X)$ is defined as follows.

Definition 3 *(lexicographical order) $A <_{lex} B$ if $min(A \triangle B) \in B$.*

That is, the lexicographical order of support vectors, or the order of natural numbers in their binary representation by sets of 1-indices.

Observation 4 *If S is a prefix of* lex, *then $|E(Q_n[S])| = f(0, |S|)$.*

Proof If $x \in S$ then all his down-neighbors are in S. Their number is the number of 1's in x. In this way, each edge spanned by S is counted exactly once in $f(0, |S|)$. ■

The solution of the dual problem is described as follows.

Theorem 5 (Harper [9]) *For every* $1 \leq m \leq 2^n$ *it holds* $f_n(m) = f(0, m)$.

Proof The inequality $f_n(m) \geq f(0, m)$ follows from the above observation. We prove the other inequality by induction on n. For $n = 1$ it clearly holds. Now, let $S \subseteq V(Q_n)$ with $|S| = m$. We split S by the first bit into $S_0 \cup S_1 = S$. Let $m_0 = |S_0|$, $m_1 = |S_1| = m - m_0$ and assume that $m_0 \geq m_1$. By induction, there are at most $f(0, m_0)$ edges in $Q_n^0[S_0]$ and at most $f(0, m_1)$ edges in $Q_n^1[S_1]$. Furthermore, there are at most m_1 edges between. Therefore,

$$\begin{aligned} |E(Q_n[S])| &\leq f(0, m_0) + f(0, m_1) + m_1 = f(0, m) - f(m_0, m_0 + m_1) + f(0, m_1) + m_1 \\ &\leq f(0, m) \end{aligned}$$

where the last step is by Lemma 2. ■

Corollary 6 *For every* $n \geq 1$, $0 < m < 2^n$ *it holds*

$$\Phi_e(m) = nm - 2 \sum_{i=0}^{m-1} h(i)$$

with optimum attained by prefixes of the lexicographical order.

Remark

- The above exact formula is inconvenient for practical purposes. It often suffices to use an estimate
$$\Phi_e(m) \geq m(n - \log_2 m)$$
with equality when $m = 2^k$ (attained by k-dimensional subcubes).

- Edges can be viewed as subcubes of dimension 1. The above results can be naturally extended for subcubes of dimension d [3]. In particular, the maximal number of d-dimensional subcubes of Q_n spanned by m vertices is
$$f_n(m, d) = \sum_{i=0}^{m-1} \binom{h(i)}{d}.$$

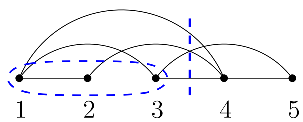
- The subgraph of Q_n induced by a prefix of the lexicographical order is sometimes [8] called an *incomplete hypercube*.

2 Linear layouts

Definition 7 *Let $G = (V, E)$ be a graph on n vertices. A* linear layout *of G is a bijection $\varphi : V \to [n]$. For a linear layout φ, $i \in [n]$, and $uv \in E$ let*

- $\theta^\varphi(i) = |\{uv \in E;\ \varphi(u) \le i \text{ and } \varphi(v) > i\}|$ — edge cut at i,
- $\delta^\varphi(i) = |\{u \in V;\ \varphi(u) \le i \text{ and } u \text{ has a neighbor } v \text{ with } \varphi(v) > i\}|$ — vertex cut at i,
- $\lambda^\varphi(uv) = |\varphi(u) - \varphi(v)|$ — length of uv.

The index φ is omitted whenever clear. Linear layouts are also called linear arrangements, linear orderings, or labelings.



$\theta(3) = 4$, $\delta(3) = 3$

Definition 8 *For a linear layout φ of $G = (V, E)$ we define the following* costs*:*

- $bw(\varphi) = \max_{uv \in E} \lambda(uv)$ — bandwidth,
- $la(\varphi) = \sum_{uv \in E} \lambda(uv)$ — linear arrangement (wirelength),
- $cw(\varphi) = \max_{i \in [n]} \theta(i)$ — cutwidth,
- $vs(\varphi) = \max_{i \in [n]} \delta(i)$ — vertex separation,
- $sc(\varphi) = \sum_{i \in [n]} \delta(i)$ — sum cut.

A *layout problem* is to find a linear layout of a given graph with the minimal cost. Thus we define the *bandwidth* $bw(G)$ of a graph G to be the minimal bandwidth over all linear layouts of G. Similarly for other costs.

Since each $uv \in E$ contributes to $\lambda(uv)$ edge cuts, linear arrangement has an alternative formula.

Observation 9 *[9] For every graph G and layout φ of G,*

$$la(\varphi) = \sum_{i \in [n]} \theta(i).$$

Remark Linear layouts can be seen as embeddings to paths and can be generalized to cycles, grids, etc.

We will see that the solution of vertex and edge isoperimetric problems in Q_n implies exact formulae for above linear layout problems in Q_n. The key idea is that the lexicographical ordering minimizes $\theta(i)$ uniformly for every $i \in [2^n]$. Similarly, the simplicial ordering, defined in the previous lecture, minimizes $\delta(i)$ uniformly.

2.1 Linear arrangement

By Observation 9 and the above remark, lexicographical ordering gives an optimal solution of the linear arrangement problem. Since every edge uv in direction $n-i$ has $\lambda^{lex}(uv) = 2^{i-1}$, we obtain

$$la(Q_n) = la(\text{lex}) = 2^{n-1}\big(1 + 2 + 4 + \cdots + 2^{n-1}\big) = 2^{n-1}(2^n - 1). \qquad [9]$$

Remark The solution of the linear arrangement problem has an interesting application. Consider the problem of finding an encoding of numbers $0, \ldots, 2^n - 1$ into binary strings of length n with minimal average error (absolute arithmetic difference after decoding back) when one bit changes. It turns out that the natural binary encoding wins!

2.2 Bandwidth

Observation 10 *For every graph G and layout φ of G,*

$$bw(\varphi) \geq vs(\varphi).$$

Proof Let $G = (V, E)$ and $n = |V|$. For every $i \in [n]$ in the set

$$\{u \in V;\ \varphi(u) \leq i \text{ and } u \text{ has a neighbor } v \text{ with } \varphi(v) > i\}$$

there is a vertex u with $\varphi(u) \leq i - \delta(i) + 1$. Since u has a neighbor v with $\varphi(v) > i$, we have $\lambda(uv) \geq \delta(i)$. The rest follows from definitions. ■

For a set $S \subseteq V$ let us define the *surface* of S by

$$\sigma S = \{v \in S \mid v \text{ has a neighbour not in } S\},$$

that is, $\sigma S = \partial_v \bar{S}$. Note that in terms of linear layouts, for the *prefix* $S = \{u \in V;\ \varphi(u) \leq i\}$ of length i we have $|\sigma S| = \delta(i)$.

In the hypercube, observe that the automorphism to antipodal vertices maps the simplicial ordering to its reverse. Thus, the prefixes of the simplicial ordering minimize also the size of surfaces uniformly for each size of the prefix. Consequently,

$$vs(Q_n) = vs(simp).$$

Finally, recall that for every prefix S of the simplicial order, σS is the suffix of S. Hence, if there is an edge between vertices u and v with labels i and j $(i < j)$, then all vertices with labels from i to $j-1$ are in the vertex cut at $j-1$. In other words, $\lambda(uv) \leq \delta(j-1)$. Therefore $bw(simp) = vs(simp)$. Altogether with Observation 10,

$$bw(Q_n) = bw(simp) = vs(simp) = vs(Q_n) = \sum_{m=0}^{n-1} \binom{m}{\lfloor m/2 \rfloor}.$$

The exact formula can be proved by induction on n [10].

2.3 Antibandwidth

Definition 11 *The* antibandwidth *of a graph G is $abw(G) = \max_{\varphi} \min_{uv \in E(G)} \lambda^{\varphi}(uv)$.*

It is known [13] that an optimal ordering for antibandwith on Q_n is, in set notation,

$$X^{(0)}, X^{(2)}, \dots X^{(1)}, X^{(3)}, \dots$$

with colexicographical order (revlex) in each level. Explicitly, we obtain

$$abw(Q_n) = 2^{n-1} - \sum_{m=0}^{n-2} \binom{m}{\lfloor m/2 \rfloor}.$$

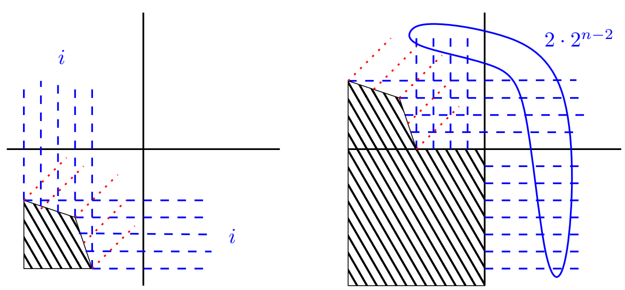
2.4 Cutwidth

As we already know, for hypercubes the optimum is attained by the lexicographic ordering [9]. Again, the antipodal automorphism of the hypercube maps lex to its reverse. Hence,

$$\theta_{lex}(i) = \theta_{lex}(2^n - i)$$

for every $0 \le i \le 2^n$. Furthermore, observe on figure below that

$$\theta_{lex}(i) = \begin{cases} 2i + \theta_{lex}^{Q_{n-2}}(i) & \text{if } 0 \le i \le 2^{n-2}, \\ 2^{n-1} + \theta_{lex}^{Q_{n-2}}(i - 2^{n-2}) & \text{if } 2^{n-2} \le i \le 2^{n-1}. \end{cases}$$



$i \le 2^{n-2}$ $\qquad$ $2^{n-2} \le i \le 2^{n-1}$

Therefore, $cw(Q_n) = 2^{n-1} + cw(Q_{n-2})$ if $n \ge 2$ and $cw(Q_0) = 0$, $cw(Q_1) = 1$. Explicitly,

$$cw(Q_n) = \begin{cases} \frac{2^{n+1}-2}{3} & \text{if } n \text{ is even}, \\ \frac{2^{n+1}-1}{3} & \text{if } n \text{ is odd}. \end{cases} \qquad [1]$$

Conjecture 12 *[4] For cyclic cutwidth (embeddings to cycles instead of paths) of hypercubes, the minimum is attained by the reflected cyclic Gray code.*

2.5 Pathwidth

Definition 13 *A* path-decomposition *of a graph $G = (V, E)$ is a sequence $X_1, \dots, X_r$ of subsets of V such that:*

(i) $\bigcup_{i=1}^{r} X_i = V$,

(ii) for every $uv \in E$ *there is* $X_i \supseteq \{u, v\}$, *and*

(iii) $X_i \cap X_k \subseteq X_j$ *for every* $i \leq j \leq k$.

The width *of a path-decomposition is* $\max_{i \in [r]} |X_i| - 1$. *The* pathwidth $pw(G)$ *of a graph* G *is the minimum width over all path-decompositions.*

The condition (iii) says that every vertex appears in a contiguous segment of the subsets (subpath). The term -1 in the definition of width is only technical, it ensures that the pathwidth of a path is 1. Intuitively, the pathwidth expresses how much the graph looks like a path.

Theorem 14 *For any graph G it holds that $pw(G) = vs(G)$.*

Proof First we show $pw(G) \geq vs(G)$. Consider an optimal path decomposition and choose an arbitrary ordering of first occurrences in X_i. Then every vertex u with first occurrence in X_i has all vertices from his vertex cut in X_i. In other words, $\delta(\varphi(u)) \leq |X_i| - 1$.

Secondly, we show $pw(G) \leq vs(G)$. Let φ be an ordering of G with minimal vertex separation. Denote u_i the vertex with $\varphi(u_i) = i$ and let $S_i = \{u_j \mid j \leq i\}$. Then for $i \in [n]$ we put $X_i = \{u_{max(1, i - vs(\varphi))}, \dots, u_i\}$ and we observe that $(X_i)_i$ form a path decomposition of width $vs(\varphi)$. ■

Corollary 15 ([12]) *For every $n \geq 1$,*

$$pw(Q_n) = bw(Q_n) = \sum_{m=0}^{n-1} \binom{m}{\lfloor m/2 \rfloor}.$$

Problem 1 *Determine the exact value of treewidth*[1] $tw(Q_n)$.

It is known [12] that $tw(Q_n) \in \Theta(2^n / \sqrt{n})$.

2.6 Book embedding (also stack layout)

Definition 16 *Let φ be a linear layout of a graph G.*

- *Edges uv and xy* cross *in φ if* $\varphi(u) < \varphi(x) < \varphi(v) < \varphi(y)$
- *A* page *(*stack*) in φ is a set of pairwise non-crossing edges.*

[1]Threewidth is more general concept than pathwidth. The difference is that sets X_i form a tree instead of a sequence (path) and it is required that for every vertex u the sets containing u form a subtree instead of a subpath.

- *A* book embedding *(*stack-layout*) of G is a linear layout of G with a partition of $E(G)$ into pages.*

- *The* page-number $pn(G)$ *(*stack-number*) is the minimum number of pages in a book embedding of G.*

Theorem 17 ([5, 11]) $pn(Q_n) = n - 1$ *for every* $n \geq 2$.

Proof Take the ordering given by the reflected Gray code and put edges of direction 1 and 2 into the same page, and let each other direction take another page. This gives an embedding into $n - 1$ pages. The proof of optimality is omitted. ■

2.7 Queue layout

Definition 18 *Let φ be a linear layout of a graph G.*

- *An edge uv* nest *an edge xy in φ if $\varphi(u) < \varphi(x) < \varphi(y) < \varphi(v)$.*

- *A* queue *in φ is a set of pairwise non-nested edges.*

- *A* queue-layout *of G is a linear layout of G with a partition of $E(G)$ into queues.*

- *The* queue-number $qn(G)$ *is the minimum number of queues in a queue layout of G.*

Conjecture 19 ([7]) $qn(Q_n) \in n - \Theta(\log n)$.

It is known [7] that $(n-2)/3 \leq qn(Q_n) \leq n - \lfloor \log_2 n \rfloor$.

Notes

For a survey on linear layouts see [6].

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