Hypercube problems

Lecture 14

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1 Preliminaries

Definition 1 *A spanning subgraph H of a graph G is a (k,t)-*detour subgraph *if $d_H(x, y) \leq d_G(x, y) + k$ for every x, y such that $d_G(x, y) \leq t$, i.e. distances in G up to t are increased in H by (a delay) at most k. Typically, we study $t = 1$ or $t = \infty$.*

Motivation: Detour subgraphs form a sparser model for communication in networks with a guaranteed distance maintained. The search for a small maximal degree is motivated by a need to get a better load distribution between all nodes.

(k,t)-detour subgraphs are also known as:

- **(2,1)-detour**: a *local detour* [4] subgraph, a *3-spanner* [6]
- **(2,∞)-detour**: a *detour* subgraph [2]
- **(k,∞)-detour**: a *k-detour* subgraph, a *k-additive* spanner [1]
- **(2,t)-detour**: a subgraph with a *t-detour property* [4]

We are interested in (k, t)-detour subgraphs of Q_n with small number of edges or small maximal degree for reasons noted above. Let us define

$$f_{k,t}(n) = min\{|E(G)|;\ \text{G is a } (k, t)\text{-detour subgraph of } Q_n\}$$

$$\Delta_{k,t}(n) = min\{\Delta(G);\ \text{G is a } (k, t)\text{-detour subgraph of } Q_n\}$$

Note: Since Q_n is bipartite, we restrict to k *even.*

2 Local detour subgraphs ((2,1)-detour) of hypercubes

Lemma 2 (Kabatiansky, Panchenko) *Q_n has a dominating set of size $\frac{2^n}{n}(1 + o(1))$.*

For proof we refer to [5]. Moreover, for $n = 2^r - 1$ where r is an integer, Q_n has perfect dominating set of size exactly $\frac{2^n}{n}(1 + \frac{1}{n})$ by a well-known Hamming code. We will use this fact in the following theorem.

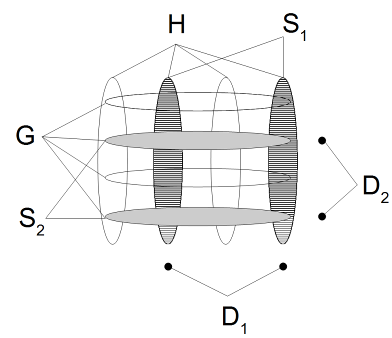
Theorem 3 [4] *$f_{2,1}(n) \leq 3 \cdot 2^n(1 + o(1))$ for every n.*

Proof Let us represent Q_n as $Q_m \,\Box\, Q_{n-m}$ for $m = \lfloor n/2 \rfloor$, where Q_m is induced by the first m and Q_{n-m} by the last $n-m$ coordinates of Q_n. Let D_1, D_2 be dominating sets of Q_m, resp. Q_{n-m} from Lemma 2. Let $S = S_1 \cup S_2$ where

$$S_1 = \{(u, v) \in V(Q_n); u \in D_1, v \in V(Q_{n-m})\},$$

$$S_2 = \{(u, v) \in V(Q_n); u \in V(Q_m), v \in D_2\}.$$

Then $|S| = (1 + o(1))\frac{2^{n+2}}{n}$.



Let G be the subgraph of Q_n spanned by all the edges incident to S. We need to show that $|E(G)| = 3.2^n(1+o(1))$ and that $E(G)$ has correct detours. The first statement can be observed from $|E(G)| = \sum_{x \in S} d_{Q_n}(x) - |E(Q_n[S])| \leq n|S| - 0.5m|S| = (3 + o(1))2^n$ where $Q_n[S]$ is the subgraph of Q_n induced by S. This comes from the fact that every vertex in S is adjacent in Q_n to at least m vertices.

What remains is to check the local detour property. An edge $\{(u, v), (u', v)\} \notin E(G)$ is detoured by $(u, v), (u, y), (u', y), (u', v)$ where $y \in D_2$ is a vertex dominating v, and $\{(u, v), (u, v')\} \notin E(G)$ is detoured by $(u, v), (x, v), (x, v'), (u, v')$ where $x \in D_1$ is a vertex dominating u. ■

Remark For $n = 2^r - 1$ (where Q_n has a perfect dominating set of vertices) one can find more precise bounds than stated in Theorem 3.

Now we show that the upper bound of the above theorem is asymptotically optimal. We may restrict this to subgraphs G with

$$|E(G)| \leq 4.2^n. \tag{1}$$

Observation 4 [4] *The number of degree-1 vertices in any (2,1)-detour subgraph G of Q_n satisfying $|E(G)| \leq 4.2^n$ is at most $8.2^n/n$.*

Proof As observed in [2], a degree-1 vertex u has a degree-n neighbour v, and v has no other degree-1 neighbours than u. This implies $\#deg_1 \leq \#deg_n \leq \frac{2|E(G)|}{n} \leq \frac{8.2^n}{n}$ by (1). ■

Notation For a subgraph G of Q_n, $\alpha \in \mathbb{R}^+$ and $v \in V(Q_n)$ let $L = L(G, \alpha) = \{u \in V(Q_n) | d_G(u) \leq \alpha\}$ (low vertices), $H = H(G, \alpha) = V(Q_n) \backslash L$ (high vertices), $d_L(v) = |N_G(v) \cap L|$, $d_H(v) = |N_G(v) \cap H|$, so $d_G(v) = d_L(v) + d_H(v)$.

Lemma 5 [4] *For any (2,1)-detour G of Q_n satisfying $|E(G)| \leq 4.2^n$, $\alpha \in \mathbb{R}^+$*

$$n|L| \leq \alpha 2^n + 2\alpha|E(G)| + \sum_{v \in H} d_L(v) d_H(v) \tag{2}$$

Proof We count oriented paths $(v_0, v_1, v_2, v_3) \in G$ with $v_0 \in L$ and $v_0 v_3 \in E(Q_n)$ – the detours. Since all missing edges must be detoured, every low vertex starts at least $n - d(v)$ detours. Hence the number is at least $\sum_{v \in L}(n - d(v))$. On the other hand, the number of oriented paths where $v_1 \in L$ or $v_2 \in L$ is at most $2|E(G)|$, because there are $2|E(G)|$ choices for (v_1, v_2). If $v_1 \in L$ or $v_2 \in L$, then there are at most α choices for v_0, v_3 as they need to be parallel in Q_n. The number of oriented 3-paths with $v_1, v_2 \in H$ is at most $\sum_{v \in H} d_L(v) d_H(v)$, since there are $d_L(v_1)$ choices for v_0, $d_H(v_1)$ choices for v_2 and v_3 is determined by v_0 and v_2. Therefore,

$$\sum_{v \in L}(n - d(v)) \leq 2\alpha|E(G)| + \sum_{v \in H} d_L(v) d_H(v),$$

which implies (2). ■

Theorem 6 [4] $f_{2,1}(n) \geq 3.2^n(1 - \sqrt{112/n})$

Proof Let us set $\alpha = \alpha(n) = \sqrt{\frac{9}{7}n}$. As

$$d_L(v) d_H(v) \leq \frac{1}{4}(d_L(v) + d_H(v))^2 = \frac{1}{4} d(v)^2 \leq \frac{n}{4} d(v),$$

Lemma 5 gives

$$\sum_{v \in H} d(v) \geq 4|L| - \frac{4\alpha}{n} 2^n - \frac{8\alpha}{n}|E(G)|.$$

By Observation 4,

$$\sum_{v \in L} d(v) \geq 2|L| - 8.2^n/n,$$

from which we get

$$2|E(G)| = \sum_{v \in V(Q_n)} d(v) \geq 6|L| - \frac{4\alpha}{n} 2^n - \frac{8\alpha}{n}|E(G)| - 8.2^n/n.$$

Because $2|E(G)| \geq \alpha|H|$, we get

$$2|E(G)|(1 + \frac{6}{\alpha} + \frac{4\alpha}{n}) \geq 6|H| + 6|L| - \frac{4\alpha}{n} 2^n - 8.2^n/n.$$

We can easily verify that

$$(1 - \frac{2\alpha}{3n} - \frac{4}{3n})/(1 + \frac{6}{\alpha} + \frac{4\alpha}{n}) \geq 1 - \frac{4\alpha}{6n} - \frac{4\alpha}{n} - \frac{6}{\alpha}.$$

Since $|H| + |L| = 2^n$, we get

$$|E(G)| \geq 3.2^n(1 - \frac{4\alpha}{6n} - \frac{4\alpha}{n} - \frac{6}{\alpha}) = 3.2^n(1 - \sqrt{112/n}).$$

■

3 Local detour subgraphs with small maximal degree

Theorem 7 [4] $\Delta_{2,1}(n) \leq 1.5\sqrt{2n} - 1$ *for every* n.

Proof For $n < 5$ the statement is obvious. Assume $n \geq 5$ and let $m = 2^r \in (\sqrt{n/2}, \sqrt{2n})$, $S = \lceil (n - m + 1)/m \rceil$ for some r. Let us partition $[n]$ into $m+1$ parts $\cup_{i=0}^{m} P_i$, where $|P_0| = m - 1$, $|P_i| \in \{s-1, s\}$ for $i = 1, ..., m$. Let H be a subgraph of Q_n on all edges of direction from P_0. Clearly, it is disjoint union of 2^{n-m+1} copies of Q_{m-1}. As $m - 1 = 2^r - 1$, Q_{m-1} has a partition into perfect dominating sets $S_1, ..., S_m$ that can be found as cosets of a Hamming code. Let S'_i denote the union of translates of S_i into all copies of Q_{m-1} in H. Then every S'_i is a dominating set in H. Now G is the union of H and all the edges of direction from P_i incident to S'_i for every $i \in [m]$.

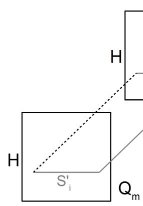


Figure 1: (2,1)–detour example

It remains to check that G is a (2,1)–detour subgraph and count $\Delta(G)$. Let us consider an arbitrary edge $(u, v) \in E(Q_n) \backslash E(G)$. The vertices u and v differ in some coordinate $i \in [n] \backslash P_0$, w.l.o.g. $i \in P_1$, which means they belong to different copies of Q_{m-1} that are adjacent in Q_n, say C_u and C_v. Then the common projection of u and v into Q_{m-1} has a neighbour z in S_1. Let z_u and z_v be the copies of z in C_u and C_v. Clearly, z_u and z_v are adjacent in Q_n. Then (u, z_u, z_v, v) is a path in G.

Finally, $\Delta(G) = m - 1 + s \leq m - 1 + n/m \leq \sqrt{n/2} + \sqrt{2n} - 1 = 1.5\sqrt{2n} - 1$. ■

Theorem 8 [4]

$$\Delta_{2,1}(n) \geq \sqrt{2n + \frac{1}{4}} - \frac{1}{2} \quad \textit{for every } n. \tag{3}$$

Proof Let G be (2,1)–detour Q_n with $\Delta(G) = \Delta_{2,1}(n)$. A detour is a path (v_0, v_1, v_2, v_3) in G such that $v_0 v_3 \in E(Q_n) \backslash E(G)$ There is at least $n2^{n-1} - |E(G)|$ such detours. Now we shall count unoriented paths (v_0, v_1, v_2) in G, there is $\sum_{v \in V(G)} \frac{1}{2} d_G(v)(d_G(v) - 1)$ of them. The number of unoriented paths (v_0, v_1, v_2) in G is at least twice the number of detours, since every detour contains two 2-edge paths and any 2-edge path in G can belong to at most one detour. Therefore #paths $\geq$ 2.#detours $= n2^n - 2|E(G)|$, where $\sum_v d_G(v) = 2|E(G)|$. Thus

$$n2^n \leq \sum_v \frac{1}{2} d_G(v)(d_G(v) + 1) \leq \frac{1}{2} 2^n \Delta(G)(\Delta(G) + 1)$$

and $(\Delta(G) + \frac{1}{2})^2 \geq 2n + \frac{1}{4}$ which implies (3). ■

4 $(2, \infty)$–detour subgraphs

Theorem 9 [2] *$f_{2,\infty}(n) < 2^n.\frac{3}{4}\sqrt{2n}$ for every n.*

Proof For $n = 1$ it is obvious. Assume $Q_n = Q_{2^k-1} \,\Box\, Q_{n-2^k+1}$, where $0 \leq 2^k - 1 \leq n \geq 1$. Q_{2^k-1} has a perfect dominating set D. Let G be a subgraph of Q_n spanned by $\{\{(u,v),(u',v)\};\ uu' \in E(Q_{2^k-1}), v \in V(Q_{n-2^k+1})\} \ \cup\ \{\{(u,v),(u,v')\}; u \in D, vv' \in E(Q_{n-2^k+1})\}$ where the first part of the union covers all copies of Q_{2^k-1} and the second part covers all copies of Q_{n-2^k+1} corresponding to vertices of D. Observe that G is a $(2,\infty)$–detour. Since G has 2^{n-k} n–degree vertices and its other vertices have degree $(2^k - 1)$, we have that $E(G) < (n2^{n-k} + 2^{n+k})/2$ and writing $x = 2^k$ we get

$$|E(G)| < (n2^{n-k} + 2^{n+k})/2 = 2^{n-1}((n/x) + x)$$

If we regard the right side as a function g of a real variable x, we can show that the graph of $g(x)$ is concave. The minimum occurs at $x' = \sqrt{n}$. We need $x = 2^k$ such that $1 \leq x \leq n+1$. Since $1 \leq \sqrt{n/2}$ for $n < 1$ and $\sqrt{2n} < n+1$ for $n > 0$, x can be chosen within factor $\sqrt{2}$ of x'. Then

$$f_{2,\infty}(n) \leq |E(G)| < g(\sqrt{2n}) = 2^n\frac{3}{4}\sqrt{2n}.$$

■

Remark The above construction preserves diameter n as proved in [2].

As for the lower bound, the following is known.

Theorem 10 [1] *$f_{2,\infty}(n) > 2^n \cdot \frac{log_2(n)}{2.10^6}$, for every n*

What is known about maximal degree in $(2,\infty)$–detour subgraphs?

Theorem 11 [1] *For every integer $k \geq 2$ and $n \geq 21$,*

$$\frac{n}{ln\ n}e^{-2k} \leq \Delta_{k,\infty}(n) \leq 20\frac{n}{ln\ n}ln\ ln\ n \qquad (4)$$

A proof of the above theorems can be found in [1].

5 (k, t)–detours for higher k

For details concerning this section read [1].

Observation 12 [4] *The construction of (2,1)–detour subgraph from the proof of Theorem 3 gives (4,∞)–detour.*

Corollary 13 [1] *For every integer $k \geq 4$, $f_{k,\infty}(n) \leq (3 + o(1))2^n$.*

This can be further generalized using the following observation.

Observation 14 [1] $f_{k+2,t}(n+1) \leq f_{k,t}(n) + 2^n$

Proof Consider the graph Q_{n+1} as the union of two copies, L and R, of Q_n joined by a perfect matching M. For each $v \in V(R)$, let $M(v)$ be the neighbour of v in L. Let G' be a (k,t)–detour graph in L with $f_{k,t}(n)$ edges. Define $E(G) = E(G') \cup M$.

To check that G is a $(k+2,t)$–detour graph in Q_{n+1}, consider arbitrary vertices x and y in Q_{n+1} at distance at most t. If both x and y are in Q, then, by definition of G', $d_G(x,y) \leq d_L(x,y) + k$. If $x \in V(L)$ and $y \in V(R)$, then

$$d_G(x,y) = 1 + d_G(x, M(y)) \leq 1 + d_L(x, M(y)) + k = d_{Q_{n+1}}(x,y) + k.$$

Finally, if both x and y are in R, then

$$d_G(x,y) = 2 + d_G(M(x), M(y)) \leq 2 + d_L(M(x), M(y)) + k = d_{Q_{n+1}}(x,y) + k + 2.$$

That concludes the proof. ■

From Observation 14, Corollary 13 and Theorem 3 we obtain the following.

Corollary 15 [1] *For every even integer* $k \geq 4$, $f_{k,\infty}(n) \leq (1 + 2^{3-k/2} + o(1))2^n$.
For every even integer $k \geq 2$, $f_{k,1}(n) \leq (1 + 2^{2-k/2} + o(1))2^n$.

6 Open problems

Conjecture 1 [2] *In a smallest (2,∞)–detour subgraph of* Q_n*, at most* $\frac{3}{4}$ *vertices have degree less than 3.*

Problem 2 [2] *Is* $\sqrt{n}2^n$ *the order of magnitude of* $f_{2,\infty}(n)$*?*

There is evidence in [2] that for small n the number of edges in any minimal $(2,\infty)$–detour subgraph and any $(2,1)$–detour subgraph is equal.

Problem 3 [2] *Is* $min\{\Delta(G) | G$ *is a spanner of* Q_n *with* $diam(G) = n\}$ *unbounded?*

It is not difficult to construct a diameter preserving subgraph of size $2^n + \binom{n}{\lfloor n/2 \rfloor} - 2$. [3]

Problem 4 [2] *Find good bounds for* $min\{|E(G)|; G$ *is spanner of* Q_n *with* $diam(G) = n\}$.

References

[1] Nana Arizumi, Peter Hamburger, and Alexandr Kostochka. On k-detour subgraphs of hypercubes. *Journal of Graph Theory*, 57(1):55–64, 2008.

[2] Pál Erdös, Peter Hamburger, Raymond E Pippert, and William D Weakley. Hypercube subgraphs with minimal detours. *Journal of Graph Theory*, 23(2):119–128, 1996.

[3] Niali Graham and Frank Harary. Changing and unchanging the diameter of a hypercube. *Discrete Applied Mathematics*, 37:265 – 274, 1992.

[4] Peter Hamburger, Alexandr V Kostochka, and Alexander Sidorenko. Hypercube subgraphs with local detours. *Journal of Graph Theory*, 30(2):101–111, 1999.

[5] Grigorii Anatol'evich Kabatiansky and V. I. Panchenko. Unit sphere packings and coverings of the hamming space. *Problemy Peredachi Informatsii*, 24(4):3–16, 1988.

[6] David Peleg and Jeffrey D Ullman. An optimal synchronizer for the hypercube. *SIAM Journal on computing*, 18(4):740–747, 1989.