Hypercube problems

Lecture 2

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1 Automorphism group

In this lecture we focus on the group of automorphisms of the hypercube. We start with some algebraic notions that will be useful. Let $G = (V, E)$ be a graph. The group of all permutations of V is called the *symmetric group on* V and is denoted by $\mathsf{Sym(V)}$.

Definition 1 *An* automorphism *of G is a bijection $f : V \to V$ such that $uv \in E$ if and only if $f(u)f(v) \in E$.*

Notice that all automorphisms of G form a group with the composition as the binary operation, the inverse automorphism as the inverse and the identity as the neutral element. This group is denoted by $\mathsf{Aut}(G)$.

Clearly, $\mathsf{Aut}(G) \le \mathsf{Sym(V)}$, i.e. automorphisms form a subgroup of the symmetric group. There is equality if and only if G is a graph without edges or G is a complete graph. Furthermore, it follows directly from the definitions that $\mathsf{Aut}(G) = \mathsf{Aut}(\overline{G})$ where $\overline{G} = (V, \overline{E})$ is the complement of G.

Definition 2 *An* action *of a group Γ on a set V is a homomorphism $\varphi : \Gamma \to \mathsf{Sym}(V)$. If the action φ is fixed, we write $g(x)$ instead of $\varphi(g)(x)$ for $g \in \Gamma$, $x \in V$ and we say that Γ* acts on *V.*

$\mathsf{Aut}(G)$ acts naturally on V by taking identity as its action $\varphi : \mathsf{Aut}(G) \to \mathsf{Sym}(V)$. In the following, this action of $\mathsf{Aut}(G)$ on V is fixed.

Definition 3 *A group Γ* acts transitively *on a set V if for every $x, y \in V$ there is $g \in \Gamma$ such that $g(x) = y$. A graph G is* vertex-transitive *if $\mathsf{Aut}(G)$ acts transitively on V.*

2 Cayley graphs

In the last lecture we saw that the hypercube can be defined as the Cayley graph $Q_n = \mathsf{Cay}(\mathbb{Z}_2^n, \{e_1, \ldots, e_n\})$. We will show that every Cayley graph is vertex-transitive (and consequently every hypercube is transitive as well).

We recall the definition of (undirected) Cayley graphs. Let Γ be a group and $S \subseteq \Gamma$ be a set such that $e \notin S$ and $s \in S \Rightarrow s^{-1} \in S$. Then the Cayley graph on Γ generated by S is

$$\mathsf{Cay}(\Gamma, S) = (\Gamma, \{xy \mid yx^{-1} \in S\}).$$

Theorem 4 *Every Cayley graph $G = \mathsf{Cay}(\Gamma, S)$ is vertex-transitive.*

Proof For each $g \in \Gamma$ (i.e. a vertex of G) we define a map $\pi_g : \Gamma \to \Gamma$ by $\pi_g : x \mapsto xg$. Note that π_g is bijective since $\pi_g(yg^{-1}) = y$ for every $y \in \Gamma$.

Let $xy \in E(G)$; that is, according to the definition, $yx^{-1} \in S$. For every $g \in \Gamma$,

$$yx^{-1} = ygg^{-1}x^{-1} = (yg)(xg)^{-1}$$

so $(yg)(xg)^{-1} \in S$, i.e. $(xg)(yg) \in E(G)$. That means that π_g is an automorphism of G.

Since $\{\pi_g \mid g \in \Gamma\}$ is closed under composition and contains identity, it forms a subgroup of $\mathsf{Aut}(G)$. Moreover, it acts transitively on Γ as for every $x, y \in \Gamma$ we have $\pi_{x^{-1}y}(x) = y$. Since $\mathsf{Aut}(G)$ has a transitive subgroup, it is transitive as well. ■

Corollary 5 *Q_n is vertex-transitive for every $n \geq 1$.*

There are many open questions regarding vertex-transitive graphs. One of the famous is the following conjecture.

Conjecture 6 (Lovász [2]) *Every connected vertex-transitive graph has a Hamilton path.*

The conjecture is open even for many subclasses of connected vertex-transitive graphs, including connected Cayley graphs. Moreover, there are only five known examples of vertex-transitive graphs that have a Hamilton path but do not have a Hamilton cycle (K_2, the Peterson graph, the Coxeter graph, and two graphs obtained from the Petersen and Coxeter graphs by replacing each vertex with a triangle [1]).

3 Orbit-stabilizer theorem

Definition 7 (stabilizer) *Let Γ be a group acting on a set V and let $x \in V$. A* stabilizer *of x is the set $\Gamma_x = \{g \in \Gamma \mid g(x) = x\}$.*

Thus, a stabilizer of x contains precisely the elements whose action fixes x. It is obvious that Γ_x is a subgroup of Γ acting on V.

Definition 8 (orbit) *Let Γ be a group acting on a set V and let $x \in V$. An* orbit *of x is the set $\Gamma x = \{g(x) \mid g \in \Gamma\}$.*

In other words, an orbit is a set of all points of V that are images of x under action of some elements from Γ. We can notice that $y \in \Gamma x$ if and only if $\Gamma_x = \Gamma_y$ (because of the existence of inverses). Thus orbits give a partition of V. In addition, it follows directly from the definition that Γ is transitive if and only if there is only one orbit in Γ.

Lemma 9 *Let $S \subseteq V$ be an orbit of Γ acting on V and let $x, y \in S$. Then $\{g \in \Gamma \mid g(x) = y\}$ is a right coset of Γ_x. Conversely, all elements in a right coset of Γ_x map x to the same point in S.*

Proof Since x, y are in the same orbit, there is $h \in \Gamma$ such that $h(x) = y$. If $g(x) = y$ for $g \in \Gamma$, then $gh^{-1}(x) = x$. Thus $gh^{-1} \in \Gamma_x$, i.e. $g \in \Gamma_x h$. That is, g is in the right coset $\Gamma_x h$.

For the other direction, let us assume that g is arbitrary from the right coset $\Gamma_x h$ for a fixed h, i.e. $gh^{-1} \in \Gamma_x$. Then $g(x) = gh^{-1}h(x) = h(x)$, where the last equality is from the fact that $gh^{-1} \in \Gamma_x$ (i.e. it stabilizes x). ■

Theorem 10 (orbit-stabilizer) *Let Γ be a group acting on a set V and let $x \in V$. Then*

$$|\Gamma_x| \cdot |\Gamma x| = |\Gamma|$$

Proof By Lemma 9, there is a one-to-one correspondence between points of the orbit Γx and the right cosets of the stabilizer Γ_x. Therefore the group Γ can be partitioned into $|\Gamma x|$ cosets, each of size $|\Gamma_x|$ which gives use the statement. ■

4 Automorphisms of hypercubes

In the rest, we study the automorphism group of hypercubes. We will define two types of automorphisms and we prove that there are no other automorphism then their compositions.

Definition 11 *For $n \geq 1$, $a \in \mathbb{Z}_2^n$, and $\pi \in S_n$ (the symmetric group on $[n]$)*

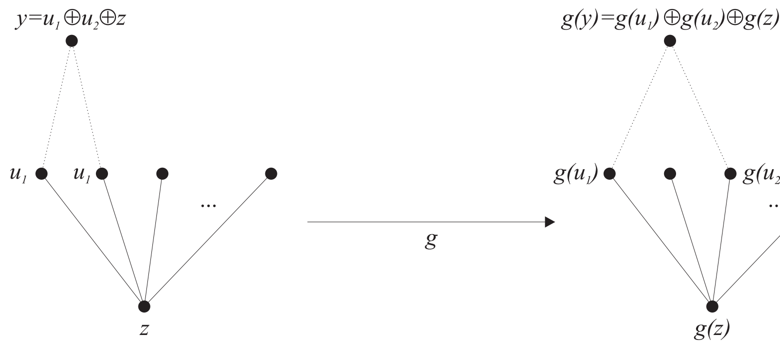
- *a* translation by a *is the map $t_a : u \mapsto u \oplus a$,*
- *a* rotation by π *is the map $r_\pi : u \mapsto u_\pi$ where $u_\pi = (u_{\pi(1)}, \ldots, u_{\pi(n)})$ for $u = (u_1, \ldots, u_n)$.*

We can view a translation as an operation that flips some fixed bits, or geometrically, it flips the hypercube. In fact, we have already encountered transitions t_a as π_a in the proof of Theorem 4. Clearly, $t_a \cdot t_b = t_{a \oplus b}$, $t_a^{-1} = t_a$ and $t_{\mathbf{0}} = id$. Therefore, $T_n = \{t_a \mid a \in \mathbb{Z}_2^n\}$ is a subgroup of $\mathsf{Aut}(Q_n)$ and $T_n \simeq \mathbb{Z}_2^n$.

As it concerns rotations, $r_\pi \cdot r_\rho = r_{\pi \cdot \rho}$, $r_\pi^{-1} = r_{\pi^{-1}}$ and $r_{id} = id$. Therefore R_n is a subgroup of $\mathsf{Aut}(Q_n)$ as well and $R_n \simeq S_n$.

Observation 12 *$\Gamma_x \simeq R_n$ for every $x \in V(Q_n)$ and $\Gamma = \mathsf{Aut}(Q_n)$.*

Proof We show that every $g \in \Gamma_x$ is uniquely determined by $g \restriction N(x)$; that is a permutation of neighbors of x extends uniquely to an automorphism of Q_n that fixes x. We proceed by levels from x, starting by level 2. Let y be in level $l \geq 2$. It has at least two neighbors u_1, u_2 in level $l - 1$. By (0, 2)-property, u_1 and u_2 have exactly one another common neighbor $z = u_1 \oplus u_2 \oplus y$ in level $l - 2$. If g is an automorphism of Q_n, it maps $g(y) = g(u_1) \oplus g(u_2) \oplus g(z)$. Thus, if g is uniquely extended to z, u_1 and u_2, its extension to y is also unique. ■



Corollary 13 $|Aut(Q_n)| = n!2^n$.

Proof Since $|\Gamma_x| = n!$ by Observation 12 and $|\Gamma x| = |V(Q_n)| = 2^n$ by Corollary 5, the statement follows from the orbit-stabilizer theorem. ■

Corollary 14 *For every $g \in Aut(Q_n)$ there is a unique $\pi \in S_n$ and a unique $a \in \mathbb{Z}_2^n$ such that $g = g_{\pi,a} = r_\pi \cdot t_a$.*

Proof It is clear that $R_n \cap T_n = \{id\}$. Furthermore, every automorphism $g \in R_n \cdot t_n \subseteq \mathsf{Aut}(Q_n)$ is defined uniquely as $g = r_\pi \cdot t_a$. By Corollary 13, there are no other automorphisms. ■

Notes

It remains to determine the structure of $Aut(Q_n)$. This will be done in the next lecture. The present lecture is (in most) covered by any textbook on algebraic graph theory. The treatment here is based on [1] which is recommended for further reading.

References

[1] C. Godsil, G. Royle, *Algebraic Graph Theory*, Springer-Verlag, New York, 2004.

[2] L. Lovász, Problem 11 in *Combinatorial Structures and their Applications*, Gordon and Breach Science Publishers, New York, 1970.