**Hypercube problems**

# Lecture 4

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In this lecture we list several characteristical properties of hypercubes.

## 1 Preliminaries

Let $G = (V, E)$ be a graph. We denote $E(A, B) := \{ab \in E \mid a \in A,\, b \in B\}$.

**Definition 1** *$G$ has* (0, 2)-property *if every two vertices share exactly* 0 *or* 2 *common neighbors. We also say that $G$ is a $(0, 2)$-graph.*

An example of a $(0, 2)$-graph other than $Q_n$ is $K_4$.

**Definition 2** *An* interval *between vertices $u$ and $v$ is*

$$I(u, v) = \{z \in V \mid d(u, z) + d(z, v) = d(u, v)\},$$

*that is, the set of vertices each of which is on some shortest $(u, v)$-path (*geodesic*).*

- Every interval in $Q_n$ forms a subcube. Specifically, $I(u, v)$ is the vertex set of $Q_n[s] \simeq Q_{d(u,v)}$ where $s = (s_1, \ldots, s_n)$ is given by

$$s_i = \begin{cases} * & \text{if } u_i \neq v_i, \\ u_i & \text{otherwise.} \end{cases}$$

**Definition 3** *$G$ is* spherical *if for every vertices $u, v$ and $x \in I(u, v)$ there is $y \in I(u, v)$ such that $I(u, v) = I(x, y)$.*

Such vertex $y$ is unique and is called the *antipodal* vertex to $x$ in $I(u, v)$.

- $Q_n$ is *spherical*. The vertex $y$ antipodal to $x$ in $Q_n[s]$, is given by

$$y_i = \begin{cases} x_i & \text{if } s_i \neq *, \\ \overline{x_i} & \text{if } s_i = *. \end{cases}$$

**Definition 4** *A set $S \subseteq V$ is* convex *if $I(u, v) \subseteq S$ for every $u, v \in S$.*

- In $Q_n$, convex sets ≈ intervals ≈ (vertex sets of) subcubes.
- There are graphs containing non-convex intervals (e. g. $K_{2,3}$).

**Definition 5** *The* antipodal projection *(antiprojection) of $x \in V$ into $S \subseteq V$ is*

$$AP(x,S) := \{y \in S \mid d(x,y) \geq d(x,s) \text{ for every } s \in S\},$$

*that is, the set of furthest vertices from $x$ in $S$.*

- In $Q_n$, any antipodal projection into any interval consists of a single vertex.

**Definition 6** *The* median set *of vertices $u, v, w \in V$ is*

$$Med(u,v,w) := I(u,v) \cap I(v,w) \cap I(u,w).$$

*A graph $G$ is* median *if $|Med(u,v,w)| = 1$ for every $u, v, w \in V$; that is, every triple of vertices has a unique median.*

Examples of median graphs: trees, grids, hypercubes. We will see in the next lecture that in some sense, they are all graphs "between" trees and hypercubes.

## 2 Characterizations of hypercubes

Let $G = (V, E)$ be a connected graph. The following statements are equivalent:

### 2.1 $G$ is a hypercube (of dimension $n$).

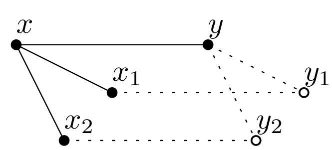
Usually, the fact that following conditions hold for hypercubes is obvious, and we just need to show the other direction (sufficiency).

### 2.2 $G$ is a $(0,2)$-graph with $|V| = 2^n$ where $n$ is the minimal degree. [9]

**Proof** Every hypercube is trivially a $(0,2)$-graph. The proof of the other direction is divided into three parts.

1. *$G$ is regular.*

   We show that every neighbor $y$ of a vertex $x$ has $deg(y) \geq deg(x)$. For every neighbor $x_i$ of $x$ other than $y$, by $(0,2)$-property, $x_i$ and $y$ have another neighbor $y_i$ other than $x$. Moreover all $y_i$'s are distinct. From symmetry we get $\deg(x) = \deg(y)$. Since $G$ is connected, it is also regular.

2. $|V| \leq 2^n$.

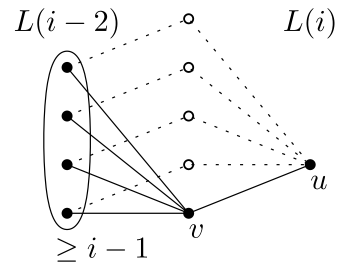
   We fix some vertex $x$ and define a level decomposition $V = L(0) \cup L(1) \cup \dots$ from $x$ by $L(i) := \{y \in V \mid d(x,y) = i\}$. It suffices to prove the following claim.

   **Claim 7** *For every $i \geq 0$ and $u \in L(i)$ it holds $|L(i)| \leq \binom{n}{i}$ and $|N(u) \cap L(i-1)| \geq i$.*

**Proof** By induction on $i$. It clearly holds for $i = 0$ and $i = 1$. Now $i \geq 2$.

The second part: every $u \in L(i)$ has a neighbor $v \in L(i-1)$. By induction for $v$ and by $(0,2)$-property, we have at least $i-1$ other neighbors of $u$ in $L(i-1)$.

The first part: from the second part we have that $i\,|L(i)| \leq |E(L(i), L(i-1))|$. Furthermore, by $n$-regularity and the second part for $L(i-1)$, we have $|E(L(i), L(i-1))| \leq (n-i+1)\,|L(i-1)|$. Hence,




$$|L(i)| \leq |L(i-1)|\,\frac{n-i+1}{i} \leq \binom{n}{i-1}\frac{n-i+1}{i} = \binom{n}{i}.$$

■

3. $|V| = 2^n$ *implies* $Q_n$.

   First, $|V| = 2^n$ implies equalities in Claim 7 for every $i$. Thus, there are no edges within levels ($G$ is bipartite) and the following property holds.

   **Property 8** *In every level decomposition, every 4-cycle intersects exactly three levels.*

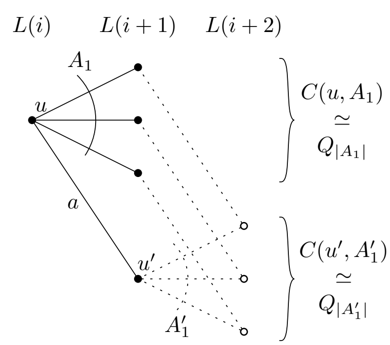
   **Definition 9** *For $u \in L(i)$ and $A \subseteq E(u, L(i+1))$ let $C(u, A)$ be the smallest $(0,2)$-subgraph of $G$ containing $A$.*

   **Claim 10** $C(u, A) \simeq Q_{|A|}$ *for every $u$ and $A$.*

   **Proof** By induction on $|A|$. It clearly holds for $|A| \in \{0, 1, 2\}$.

   Otherwise, let $A = A_1 \cup \{uu'\}$. By $(0,2)$-property we have a set of corresponding edges $A_1'$ from $u'$ (see figure on the right). By induction, we have disjoint subcubes $C(u, A_1) \simeq Q_{|A|-1}$ and $C(u', A_1') \simeq Q_{|A|-1}$.

   Moreover, by inductively using the $(0,2)$-property we find a perfect matching between corresponding vertices of the subcubes, which gives us $Q_{|A|}$. ■




■

## 2.2' $G$ is a $(0,2)$-graph with Property 8. [3]

**Proof** Property 8 implies Claim 7 with equalities for every $i$ and we repeat the rest of the proof for (2.2) to imply $Q_n$. ■

## 2.3 $G$ is $n$-regular, $|V| = 2^n$, and $E(N(u), N(v))$ is a perfect matching for every edge $uv$. [14]

**Proof** The last condition implies $(0, 2)$-property. Apply condition (2.2). ■

## 2.4 $G$ is bipartite, the number of shortest $uv$-paths is $d(u, v)!$ for every $u, v \in V$. [5]

**Proof** The condition implies $(0, 2)$-property. Furthermore, bipartiteness implies Claim 7 with equalities for every $i$, hence $|V| = 2^n$. We finish the proof by applying (2.2). ■

## 2.5 $|AP(x, I(u, v))| = 1$ for every $x, u, v \in V$. [3]

That is, antiprojections into intervals are unique.
**Proof**

1. *$G$ is bipartite.*

   By contradiction, consider a shortest odd cycle $C$, select $x$ arbitrarily on $C$ and $uv$ as the "opposite" edge to $x$ on $C$. Then $AP(x, I(u, v)) = \{u, v\}$.

2. *$G$ is a $(0, 2)$-graph.*

   For $ux, xv \in E$ there is exactly one common neighbor of $u$ and $v$ other than $x$, otherwise $|AP(x, I(u, v))| \neq 1$.

3. *Property 8 holds.*

   Suppose there is a 4-cycle $(u, a, v, b)$ in a level decomposition from $x$ with $u, v \in L(i)$, $a, b \in L(i-1)$ for some $i$. Since $G$ is bipartite and $(0, 2)$-graph, we have $I(u, v) = \{u, v, a, b\}$. Then we obtain a contradiction $|AP(x, I(u, v))| = \{u, v\}$.

We finish the proof by applying condition (2.2'). ■

## 2.5' $G$ is $K_{2,3}$-free and $|AP(x, C)| = 1$ for every $x \in V$ and a convex set $C \subseteq V$. [3]

**Proof**

1. *$G$ is bipartite.* As in the previous condition.

2. *$G$ is a $(0, 2)$-graph.*

   If $ux, xv \in E$, there exists another common neighbour $w$ of $u$ and $v$, otherwise $|AP(x, \{u, x, v\})| \neq 1$. A third common neighbour would form $K_{2,3}$ ($\{u, v\}$ as one partite, their common neighbours as the other one).

3. *Property 8 holds.* Since 4-cycles in a $(0, 2)$-graph are convex, we can proceed as in the previous condition.

We finish the proof by applying condition (2.2'). ■

### 2.6 $G$ is bipartite and every interval induces a $(0,2)$-graph. [11]

**Proof**

1. *$G$ is $(0,2)$-graph.* If $ux, xv \in E$, then $I(u,v)$ is a 4-cycle (the only $(0,2)$-supergraph).
2. *Property 8 holds.* Otherwise, an interval from $x$ (the starting vertex of a level decomposition) could contain only three vertices from some 4-cycle, which would not induce a $(0,2)$-graph.

We finish the proof by applying condition (2.2'). ■

### 2.7 $|V| = 2^n$ and $Q_0, Q_1, \ldots, Q_{n-1}, G$ are the all nonisomorphic convex subgraphs of $G$. [15]

**Proof** $P_3$, $C_3$, and $K_{2,3}$ are not convex subgraphs of $G$, so it is a $(0,2)$-graph. We finish the proof by applying condition (2.2). ■

### 2.8 $G$ is bipartite and spherical. [2, 16]

Proof omitted.

### 2.8' $G$ is triangle-free and spherical. [8]

Proof omitted as well.

### 2.9 $G$ is bipartite and *interval-regular*. [12]

**Definition 11** *$G$ is* interval-regular *if $|N(u) \cap I(u,v)| = d(u,v)$ for every $u, v \in V(G)$.*

**Proof**

1. *$G$ is $(0,2)$-graph.* If $ux, xv \in E$, then $d(u,v) = 2$ (since $G$ is bipartite). By $|N(u) \cap I(u,v)| = 2$, there is exactly one common neighbor of $u$ and $v$ other than $x$.
2. *$G$ is $n$-regular (for some $n$).* As in condition (2.2) step 1.
3. *$V = 2^n$.*

   Take a level decomposition from $x$. Again, we show Claim 7 with equalities using the assumption that every $u \in L(i)$ has exactly $i$ neighbors in $L(i-1)$ by interval regularity for $I(u,x)$. Thus, every $u \in L(i-1)$ is left with $n-(i-1)$ neighbors in $L(i)$ and every $v \in L(i)$ has $i$ neighbors in $L(i-1)$. By induction, we have

   $$|L(i)| = |L(i-1)| \, \frac{n-(i-1)}{i} = \binom{n}{i-1} \frac{n-(i-1)}{i} = \binom{n}{i}.$$

   Consequently, $\sum_i |L(i)| = 2^n$.

We finish the proof by applying condition (2.2) (only step 3 is needed). ■

The following characterizations are listed for completeness, proofs are omitted.

### 2.10 $G$ is bipartite, antipodal, and $(0,2)$-graph. [13]

**Definition 12** *$G$ is* antipodal *if for every vertex $u$ there is a vertex $v$ such that $I(u,v) = V$.*

Note that the vertex $v$ is unique since we cannot have at the same time a shortest $(u,v)$-path through $v'$ and a shortest $(u,v')$-path through $v$.

### 2.11 $G$ is distance monotone, interval monotone, and ($\delta(G) \geq 3$ or $G \in \{Q_0, Q_1, Q_2\}$). [4]

**Definition 13**

- *An interval $I$ is* closed *if for every $w \in V \setminus I$ there is $w' \in I$ with $d(w,w') > \mathrm{diam}(I)$.*
- *$G$ is* distance monotone *if every interval in $G$ is closed.*
- *$G$ is* interval monotone *if every interval in $G$ is convex.*

### 2.12 $G$ is bipartite, interval distance monotone, and ($\delta(G) \geq 3$ or $G \in \{Q_0, Q_1, Q_2\}$). [1]

**Definition 14** *$G$ is* interval distance monotone *if every subgraph induced by interval in $G$ is distance monotone.*

Interval distance monotonicity is trivially implied by distance monotonicity, but not vice-versa (consider e. g. $C_{2k+1}$). The proof of the following characterizations will follow from the next lecture.

### 2.13 $G$ is $n$-regular and median. [11]

### 2.14 $G$ is median and $|W_{uv}| = |V|/2$ for every $uv \in E$.

**Definition 15** *For an edge $uv$ we denote*

$$W_{uv} := \{ x \in V \mid d(x,u) < d(x,v) \} .$$

### 2.15 $G$ is median and $W_{uv} = U_{uv}$ for every $uv \in E$.

**Definition 16** *For an edge $uv$ we denote*

$$U_{uv} := \{ x \in W_{uv} \mid x \textit{ has a neighbour in } W_{vu} \} .$$

### 2.16 $G$ is median and for every $ab, uv \in E$, if $W_{ab}$ and $W_{uv}$ are disjoint, then $W_{ab} \cup W_{uv} = V$. [7]

## Problems

1. Determine whether *bipartite* can be weakened to *triangle-free* (or another weaker condition) in the above characterizations.

2. Find “completely” new characterizations of hypercubes.

## Notes

Characterizations (2.14) and (2.15) are proved in [6].

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