Hypercube problems

Lecture 5

November 6, 2012

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1 Partial cubes

A subgraph H of G is *isometric* if $d_H(u,v) = d_G(u,v)$ for every $u,v \in V(H)$; that is, H preserves all distances from G. Clearly, every isometric subgraph is an induced subgraph. To see that the converse implication is not true, consider a path P_4 in the cycle C_5.

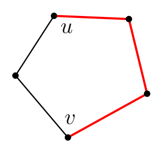

Figure 1: A path P_4 in the cycle C_5 is induced but not isometric subgraph.

A graph G is a *partial cube* if it has an isometric embedding into Q_n for some n. That is, G is isomorphic to an isometric subgraph of Q_n for some n. The smallest such n (if it exists) is the *isometric dimesion* of G, denoted by $dim_I(G)$.

Observation 1

- *$dim_I(T) = n-1$ for every tree T on n vertices.*
- *$dim_I(C_{2n}) = n$ for every $n \geq 2$, with isometric embedding such that every direction is used exactly twice, on antipodal edges.*

Examples of partial cubes: the Desargues graph, the permutahedron, benzenoid graphs, linear extension graphs, hyperplane arrangement graphs, Fibonacci cubes, see Figure 2.

2 Djoković-Winkler relation

Let $G = (V,E)$ be a bipartite graph. For $uv \in E$ let $W_{uv} = \{x \in V; d(x,u) < d(x,v)\}$ and $W_{vu} = \{x \in V; d(x,v) < d(x,u)\}$. Since G is bipartite, $W_{uv} \cup W_{vu}$ is a partition of V.

Observation 2 *All vertices on every shortest xu-path are in W_{uv} (that is, $I(x,u) \subseteq W_{uv}$) if $x \in W_{uv}$.*

The Djoković-Winkler[1] relation, denoted by Θ, is defined as follows.

Definition 3 *$e\Theta f$ for edges $e = uv$ and f if f joins a vertex in W_{uv} with a vertex in W_{vu}.*

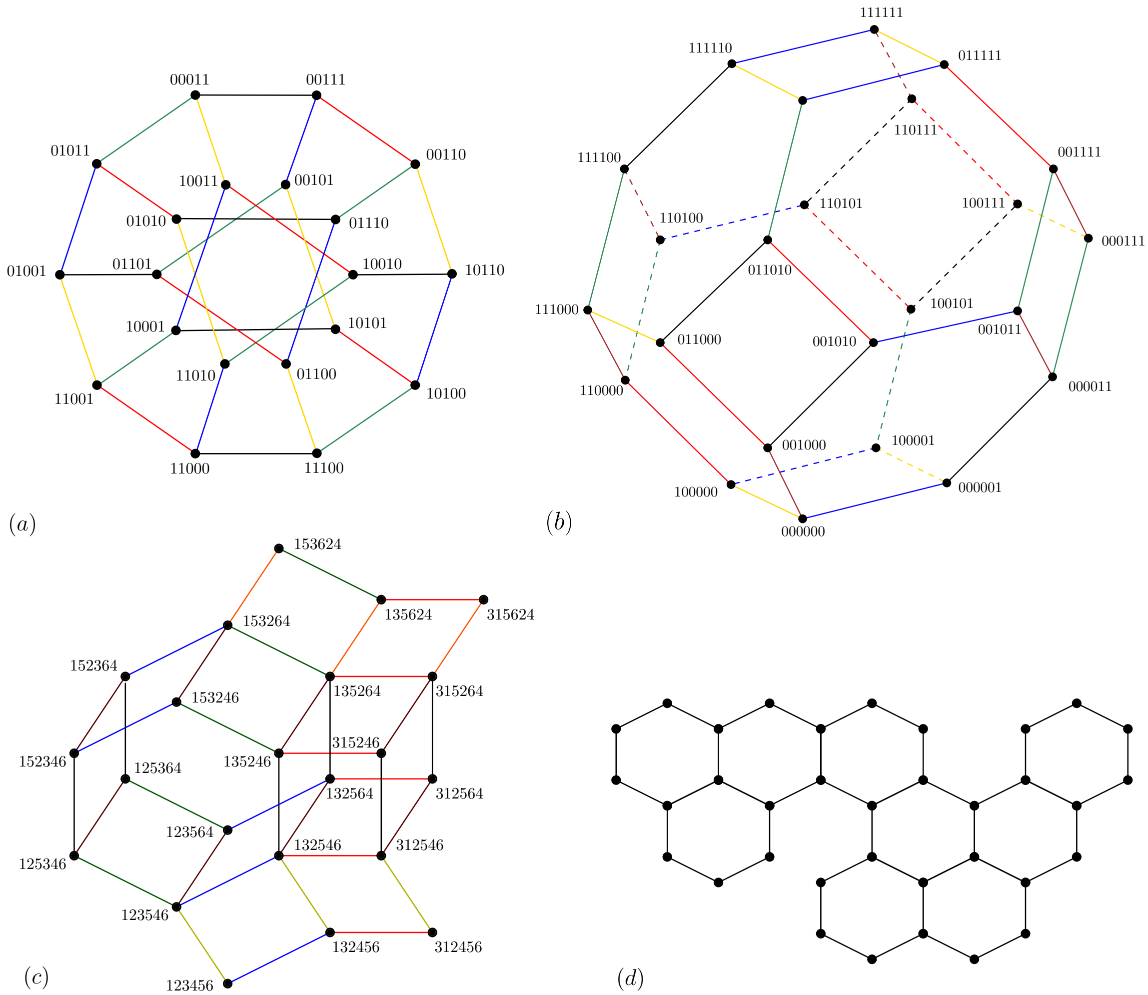

Figure 2: Examples of partial cubes: (a) the Desargues graph (also known as the generalized Petersen graph $P(10,3)$ or the middle level graph of Q_5), (b) the permutahedron of order 4 (also known as the truncated octahedron), (c) the linear extension graph of a (now secret) poset, (d) a benzenoid graph.

From Observation 2 we obtain the following.

Observation 4 *For every bipartite graph G,*

- *the relation Θ is reflexive and symmetric,*
- *if edges e, f both belong to a same shortest path in G, then $e \not\Theta f$.*

To see that the relation Θ does not have to be transitive, consider $K_{2,3}$ on Figure 3(b).

[1] The definition given here is due to Djoković [8]. Winkler [17] defined another relation which in bipartite graphs coincides with this definition. Hence it is commonly called Djoković-Winkler relation.

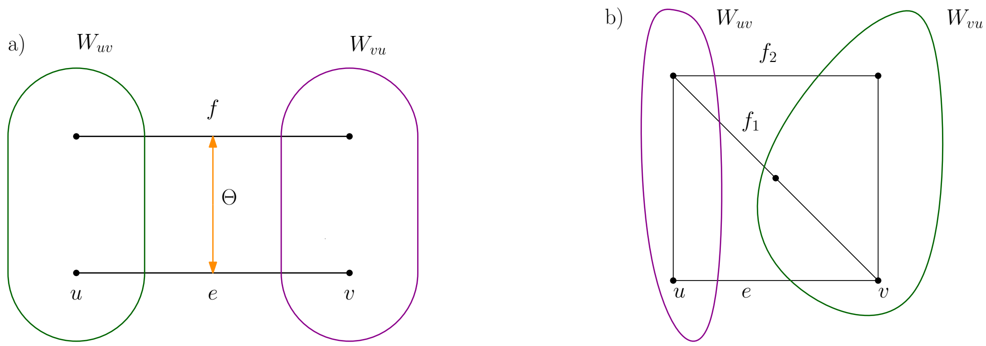

Figure 3: (a) a partition of V to W_{uv} and W_{vu}, (b) $K_{2,3}$ with $e\Theta f_1$, $e\Theta f_2$, but $f_1 \not\Theta f_2$.

3 Characterization of partial cubes

A set $S \subseteq V$ is *convex* if $I(u,v) \subseteq S$ for every $u,v \in S$ where $I(u,v)$ is the interval between u and v. That is, with every two vertices S contains every shortest path between them.

Theorem 5 (Djoković [8], Winkler [17]) *Let $G = (V,E)$ be a connected graph. The following statements are equivalent.*

1. *G is a partial cube,*
2. *G is bipartite and W_{uv}, W_{vu} are convex for every $uv \in E$,*
3. *G is bipartite and Θ is transitive (and consequently, an equivalence).*

Proof

- 1) $\Rightarrow$ 2) Let $f : V \to V(Q_n)$ be an isometric embedding. The subgraph $f(G)$ of Q_n is clearly bipartite which implies that G is bipartite.

  If $uv \in E$ then clearly $f(u) \oplus f(v) = e_i$ for some i. Assume without loss of generality that $f(u)_i = 0$. Since $f(G)$ is an isometric subgraph of Q_n and distances in Q_n are measured by the Hamming distance it follows that $f(x)_i = 0$ for every $x \in W_{uv}$. A shortest path between $f(x_1)$, $f(x_2)$ for $x_1, x_2 \in W_{uv}$ clearly contains at most one edge of each direction. Thus, it does not contain any edge in the direction i and all vertices on the shortest path have the i-th coordinate equal to 0. This means that the shortest path between $f(x_1)$ and $f(x_2)$ stays in $f(W_{uv})$ so the same is true for its preimage - the shortest path between $x_1, x_2 \in W_{uv}$ in G.

- 2) $\Rightarrow$ 3) Let $(uv)\Theta(xy)$, say $x \in W_{uv}$ and $y \in W_{vu}$. Symmetry of Θ implies that $u \in W_{xy}$ and $v \in W_{yx}$. We claim that $W_{uv} = W_{xy}$. Suppose $w \in W_{uv} \setminus W_{xy}$. Then u is on a shortest wv-path and $w \in W_{yx}$. The convexity of W_{yx} and the fact that $w, v \in W_{yx}$ imply that $u \in W_{yx}$, a contradiction. The claim implies that Θ is transitive.

- 3) $\Rightarrow$ 1) We prove this implication by constructing an isometric embedding f of G to Q_n where n is the index of Θ, i.e. $n = |E \,/\, \Theta|$. Let $e_i = u_i v_i$ be a representative of the i-th class of Θ. The embedding $f : V(G) \to \mathbb{Z}_2^n$ is defined as follows.

$$f(x)_i = \begin{cases} 0 & \text{if } x \in W_{u_i v_i} \\ 1 & \text{if } x \in W_{v_i u_i} \end{cases}$$

Note that $e \Theta e'$ if and only if $f(e)$ and $f(e')$ are of the same direction in Q_n.

For distinct $x, y \in V$ a shortest xy-path has edges from distinct Θ-classes which means that $f(x) \neq f(y)$, so f is injective. By the same reason, f maps shortest paths from G to shortest paths in Q_n. Thus, f is an isometric embedding.

■

Note that from the proof it follows that $dim_I(G)$ is the index of Θ. Since the relation Θ can be found and tested for transitivity in a polynomial time, we obtain the following.

Corollary 6 *Partial cubes can be recognized in polynomial time.*

In fact, they can be recognized in $O(mn)$ time and $O(n^2)$ space where $n = |V|$ and $m = |E|$ [1].

4 Median graphs

A *set of medians* for a triple of vertices x, y, z is defined by

$$Med(x, y, z) = I(x, y) \cap I(y, z) \cap I(z, x).$$

A graph G is a *median* graph if $|Med(x, y, z)| = 1$ for every $x, y, z \in V(G)$. That is, every triple has a unique median, which is then denoted by $med(x, y, z)$.

The class of median graphs includes all trees, grids, hypercubes (by the following proposition). We will see that median graphs are in some sense graphs "between" trees and hypercubes. They occur in many at first seemingly unrelated areas such as the study of solutions of 2-SAT [10], stable configurations of nonexpansive networks [10], stable matchings in the well-known stable roommate problem [7], median semilattices [4].

Proposition 7 *Q_n is a median graph for every $n \geq 1$ with $med(x, y, z) = maj(x, y, z)$ where maj is the coordinate-wise majority function.*

Proof If two vertices agree in a coordinate, say $x_i = y_i$, then every $m \in Med(x, y, z)$ has $m_i = x_i = y_i$ since m belongs to a shortest xy-path, so $m = maj(x, y, z)$. On the other hand, if $m = maj(x, y, z)$ then $x_i = y_i$ implies $m_i = x_i = y_i$, so m lies on a shortest path between every pair of vertices. ■

Every median graph is bipartite. Indeed, for a contradiction consider a shortest odd cycle with a vertex x antipodal to an edge yz. Then $Med(x, y, z) = \emptyset$.

Theorem 8 *Every median graph $G = (V, E)$ is a partial cube.*

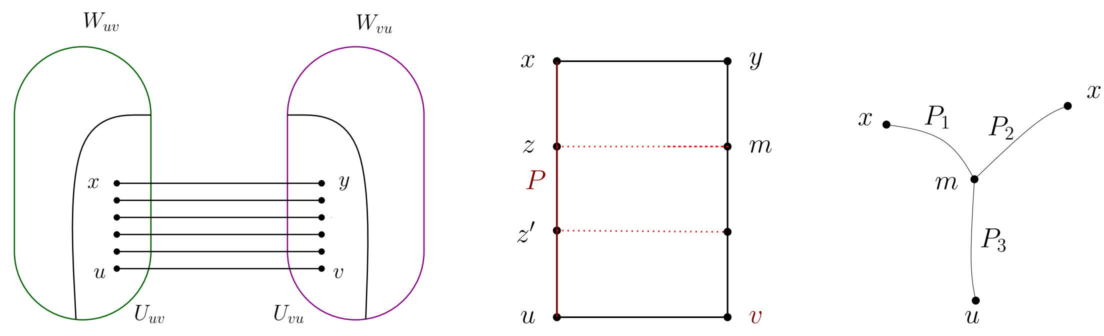

Figure 4: (a) The sets U_{uv} and U_{vu}, (b) a shortest xu-path P in U_{uv} and adjacent vertices from U_{vu}, (c) the shortest paths between x and y, x and u, y and u.

Proof For $uv \in E$ let $U_{uv} = \{x \in W_{uv} \mid x \text{ has a neighbor in } W_{vu}\}$ and similarly U_{vu}.

Claim 9 *If $x \in U_{uv}$ then every shortest xu-path belongs to U_{uv} (that is, $I(x,u) \subseteq U_{uv}$).*

Let z be a neighbor of x on a shortest xu-path P, see Figure 4(b). Clearly, $d(z,y) = 2$ and $d(z,v) = d(x,u) = d(y,v)$. Then $m = med(z,y,v)$ must be a neighbor of z lying on a shortest yv-path. Therefore $z \in U_{uv}$. Then we continue this argument for the next vertex z' on P until we reach u.

Claim 10 *U_{uv} induces an isometric subgraph.*

Let $x, y \in U_{uv}$ and $m = med(x,y,u)$. Let P_1 be a shortest xm-path, P_2 a shortest ym-path and P_3 a shortest um-path, see Figure 4(c). By Claim 4, P_1P_3 and P_2P_3 are in U_{uv} which means that P_1P_2 also lies in U_{uv} (i.e. $d_{\langle U_{uv}\rangle}(x,y) = d_G(x,y)$).

Therefore, W_{uv} and similarly W_{vu} are convex and we may apply Theorem 5. ■

Not every partial cube is a median graph, see Figure 5.

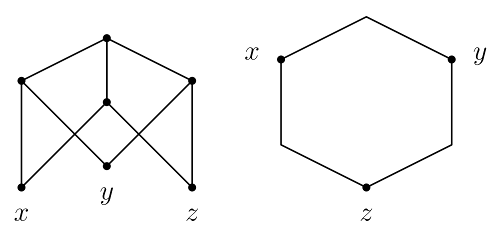

Figure 5: Two examples of partial cubes that are not median graphs

5 Mulder's convex expansion

Let $G = (V,E)$ be a connected graph and $S \subseteq V$ be a convex set. The expanded graph $G' = exp(G,S)$ is obtained as follows:

- for every vertex u in S add a copy u' and join u and u' by an edge,

- insert an edge $u'v'$ between new vertices whenever uv is an edge of $\langle S \rangle$.

The set of new vertices in denoted by S' (a copy of S), see Figure 6.

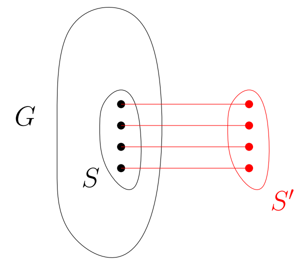

Figure 6: The expansion procedure.

Since a subgraph of a median graph induced by a convex set is clearly median, by considering all cases for triples in G' it can be shown that if G is a median graph, then G' is also a median graph.

Interestingly, every median graph can be obtained by this procedure[2] starting from a single vertex. This shows us a "tree-like" structure of median graphs.

Theorem 11 (Mulder [14]) *A graph G is a median graph if and only if it can be obtained from a single vertex by a sequence of convex expansions.*

For a proof we refer to [14].

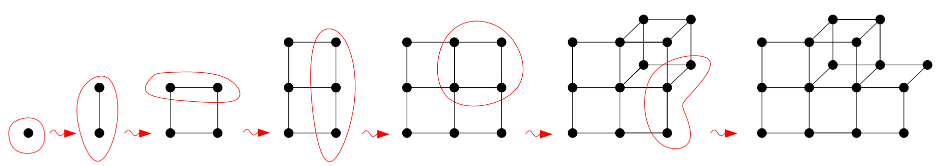

Figure 7: Examples of convex expansions.

6 Euler-type formula

Recall that vertices and edges can be seen as 0-faces and 1-faces, respectively. Thus the following relation can been regarded as an Euler-type formula.

Theorem 12 (Škrekovski [16]) *Let G be a median graph, $k = dim_I(G)$, and let q_i be the number of subcubes of dimension i in G. Then,*

$$\sum_{i \geq 0} (-1)^i q_i = 1 \text{ and } k = -\sum_{i \geq 0} (-1)^i i q_i.$$

[2]Originally, the convex expansion is described in a more general way. However, it can be shown that this simplified version suffices.

Proof Let $G = exp(G', S)$, q'_i be the number of copies of Q_i in G', q^S_i be the number of copies of Q_i in $\langle S \rangle$. From the expansion procedure, observe that $q_i = q'_i + q^S_i + q^S_{i-1}$ for every $i \geq 0$ (where we define $q^S_{-1} = 0$). Then,

$$\sum_{i \geq 0} (-1)^i q_i = \sum_{i \geq 0} (-1)^i q'_i + \underbrace{\sum_{i \geq 0} (-1)^i q^S_i + \sum_{i \geq -1} (-1)^{i+1} q^S_i}_{= 0 \text{ since } q^S_{-1} = 0} = 1 \tag{1}$$

by induction for G'. Furthermore, applying induction for G' and (1) for $\langle S \rangle$ gives us

$$\begin{aligned} k = k' + 1 &= -\sum_{i \geq 0} (-1)^i i q'_i + \sum_{i \geq 0} (-1)^i q^S_i = \\ &= -\sum_{i \geq 0} (-1)^i i q'_i - \sum_{i \geq 0} (-1)^i (i q^S_i - (i+1) q^S_i) = \\ &= -\sum_{i \geq 0} (-1)^i i q'_i - \sum_{i \geq 0} (-1)^i i q^S_i - \sum_{i \geq 1} (-1)^i i q^S_{i-1} = -\sum_{i \geq 0} (-1)^i q_i. \end{aligned}$$

■

7 Open problems

Problem 1 (Eppstein [9]) *Find a nonplanar cubic partial cube other than the Desargues graph.*

(Here, cubic means 3-regular.)

Conjecture 13 (Brešar et al. [5]) *Every cubic partial cube is Hamiltonian.*

Problem 2 *Characterize Hamiltonian partial cubes (median graphs).*

A partial cube $G = (V, E)$ is Θ-*graceful* if there is a bijection $f : V \to \{0, \ldots, |V| - 1\}$ such that for every $e_1, e_2 \in E$:

$$e_1 \Theta e_2 \text{ if and only if } f'(e_1) = f'(e_2)$$

where $f'(uv) = |f(u) - f(v)|$.

Conjecture 14 (Brešar, Klavžar [6]) *Every partial cube is* Θ*-graceful.*

Remark This conjecture implies the graceful tree conjecture.

8 Notes

Median graphs were first studied by Avann [2], Nebeský [15], and independently by Mulder [14]. There is an extensive literature on median graphs, see [3, 13] for surveys. A good starting reference is also the book[3] of Imrich and Klavžar [12].

[3]The second edition [11] is already available.

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