Hypercube problems

Lecture 6

November 14, 2012

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1 Nonexpansive maps

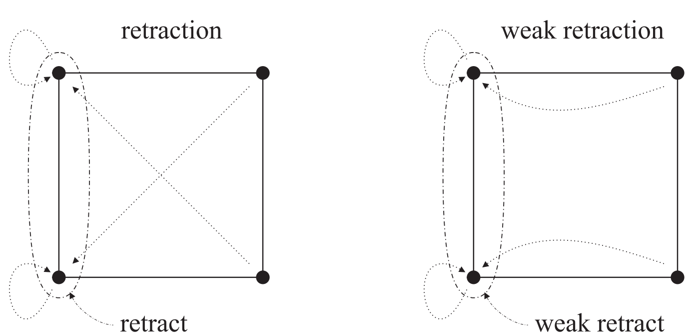
Definition 1 (nonexpansive map) *Let G and H be graphs. A mapping $f : V(G) \to V(H)$ is called* nonexpansive *if $d_H(f(u), f(v)) \leq d_G(u, v)$ for every $x, y \in V(G)$ (i.e. it does not increase distances).*

Nonexpansive maps are also known as *weak homomorphisms*, mappings satisfying the property that for every $xy \in E(G) : f(x)f(y) \in E(H)$ or $f(x) = f(y)$. Equivalence of these two notions is easy to see, as weak homomorphism can not prolong any path and vice-versa, nonexpansive maps do not increase distance of adjacent vertices.

A nonexpansive map of G to G is called simply a *nonexpansive map on G*. It is easy to see that composition of two nonexpansive mappings is also nonexpansive and that image $f(G)$ of a connected graph G under a nonexpansive map f is also connected.

Definition 2 (fixed point, periodic point) *Let f be a (nonexpansive) map on G. We say that x is a* fixed point *of f if $f(x) = x$. We denote the set of all fixed points of f by $\mathsf{fix}(f)$. We say that x is a* periodic point *of f if $f^{(p)}(x) = x$ for some $p \geq 1$. The smallest such p is the* period *of x. We denote the set of all periodic points by $\mathsf{per}(f)$.*

Definition 3 (retraction, weak retraction) *A* retraction (weak retraction) *on G is an endomorphism (nonexpansive map, respectively) $f : V(G) \to V(G)$ such that $f^{(2)}(x) = f(x)$ for every x (this property is called* idempotence*). In other words, $\mathsf{rng}(f) = \mathsf{fix}(f) = \mathsf{per}(f)$. A* (weak) retract *is a subgraph induced by $\mathsf{rng}(f)$ of a (weak) retraction f.*



Clearly, every retract is also a weak retract. But the converse is not true, as the single vertex K_1 is a weak retract of any graph, but is a retract only of graphs without edges. On the other hand, in hypercubes weak retracts are also retracts (up to the mentioned case of K_1).

Observation 4 *Weak retracts (and thus retracts as well) are isometric subgraphs.*

Lemma 5 *Let f be a nonexpansive map on $G = (V, E)$. Then $f^{(|V|!)}$ is a weak retraction with the corresponding weak retract induced by $\mathsf{per}(f)$.*

Proof Let us denote $g := f^{(|V|!)}$ and let x be a periodic point of f, i.e. $x \in \mathsf{per}(f)$. Then x is a fixed point of g since the period p of x divides $|V|!$. Moreover, for every $x \in V$ and $k \geq |V| - 1$ it holds that $f^{(k)}(x) \in \mathsf{per}(f)$, which gives us that $g(x)$ is also a periodic point of f. As a consequence, we have that $\mathsf{fix}(g) = \mathsf{rng}(g) = \mathsf{per}(f)$ and therefore g is a weak retraction on $\mathsf{per}(f)$. ■

As we know that periodic points induce a weak retract and weak retractions preserve connectivity, we have following corollary.

Corollary 6 *Let f be a nonexpansive map on a connected graph G. Then $\mathsf{per}(f)$ induces a connected graph.*

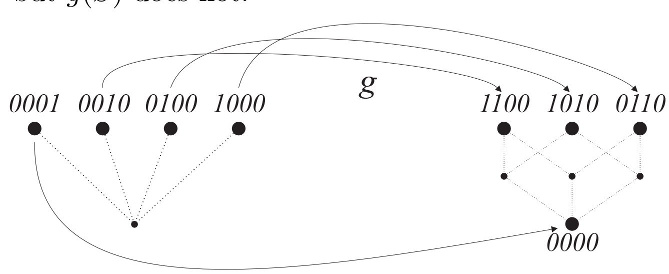
2 Nonexpansive maps on hypercubes

We will need the following simple but useful lemma.

Lemma 7 *Let $S \subseteq V(Q_n)$ be a set of vertices inducing a connected subgraph of Q_n and let $g : S \to V(Q_n)$ be a mapping such that $d_H(g(x), g(y)) = d_H(x, y)$ for every $x, y \in S$ (d_H denotes the Hamming distance). Then g can be extended into an automorphism g' of Q_n.*

Proof Let us fix some $x \in S$. We find a permutation $\pi \in S_n$ such that $g(x \oplus z) = g(x) \oplus z_\pi$ holds for every $x \oplus z \in S$. In order to construct such π we will proceed by levels according to the distance from x and extend partial injection $\pi' \subseteq [n] \times [n]$ as needed. We start with $\pi' = \emptyset$. If $y' = y \oplus e_i$ for y' on a current level and y on the previous level so that $g(y') = g(y) \oplus e_j$, then we add (i, j) to π'. This construction will never violate injectivity since g preserves distances and the set S is connected. Nevertheless, the final π' do not have to be permutation but since it is injective, it can be extended to a permutation π. The desired automorphism that extends g will be $g' = r_\pi t_{x_\pi \oplus g(x)}$ (notice that $t_{x_\pi \oplus g(x)}$ is an automorphism that maps x onto $g(x)$ as we discussed in previous lectures). Clearly $g' \restriction S = g$. ■

To illustrate that the connectness of the set S is necessary in the previous lemma, we will consider this example in Q_4: $S = \{0001, 0010, 0100, 1000\}$ and the mapping g which maps elements of S to 0000, 1100, 1010, 0110 in respective order (see the illustration below). In this situation there does not exist any extension of g to an automorphism since S has a common neighbour but $g(S)$ does not.



Theorem 8 *Let f be a nonexpansive map on Q_n. There is $g \in \mathsf{Aut}(Q_n)$ such that $f(x) = g(x)$ for every $x \in \mathsf{per}(f)$. In addition, $g \restriction \mathsf{per}(f)$ is an automorphism of the graph induced by $\mathsf{per}(f)$.*

Proof By Lemma 5 we know that $\mathsf{per}(f)$ induces a weak retract and we know that weak retracts are connected. Moreover, we have that $d_H(f(x), f(y)) = d_H(x, y)$ for every $x, y \in \mathsf{per}(f)$. The first part of the statement follows directly from Lemma 7.

The second part is an easy consequence of the fact that for every periodic point x of f we have $f(x) = g(x) \in \mathsf{per}(f)$. ■

3 Median graphs (revisited)

Let us recall that G is a *median graph* if for every triple of vertices x, y, z the median set (i.e. the intersection of intervals between all pairs) is a singleton, precisely $|I(x, y) \cap I(x, z) \cap I(y, z)| = 1$. Then their median is denoted by $med(x, y, z)$. We already know that hypercube is a median graph.

Lemma 9 *Let f be a nonexpansive map on a median graph G and let x, y, z be some fixed points of f. Then $med(x, y, z)$ is a fixed point of f as well.*

Proof Let m denote the median of x, y, z. We consider the following chain of inequalities:

$$\begin{aligned} d_H(x, y) &\le d_H(x, f(m)) + d_H(y, f(m)) = d_H(f(x), f(m)) + d_H(f(y), f(m)) \\ &\le d_H(x, m) + d_H(y, m) = d(x, y) \end{aligned}$$

where the first inequality is a triangle inequality for the metric d_H. The next equality follows from the fact that x and y are fixed points of f. The subsequent inequality holds since f is nonexpansive. And the last equality holds as m is a median of x and y. As the last items equals the first one all inequalities must be in fact equalities and therefore $f(m)$ is a median of x, y and z, and consequently, $m = f(m)$ is a fixed point of f. ■

Remark The statement of Lemma 9 holds even if we replace $\mathsf{fix}(f)$ with $\mathsf{per}(f)$.

4 Median sets

Definition 10 *We say that $S \subseteq V(Q_n)$ is a* median set *if it is closed under the operation med (that is, the majority function).*

Thus Lemma 9 states that $\mathsf{fix}(f)$ of a nonexpansive f on Q_n is a median set.

Observation 11 *Every connected median set in Q_n induces a median graph.*

Corollary 12 (of Lemma 9) *Every weak retract of a median graph is a median graph. And every weak retract of Q_n is induced by a connected median set.*

In hypercubes we have correspondences presented in the following diagram. Some of them are already shown, some of them will be shown in the rest of this text and the following lecture. The symbol $*$ represents the correspondence that every median graph has an isometric embedding to a connected median set in a hypercube.

a retract of the hypercube	$\overset{\text{is}}{\Longrightarrow}$	a weak retract of the hypercube		
induces $\Uparrow$ (if not K_1)		is induced by $\Downarrow$		
a set of solutions of a 2-CNF with no equivavalent variables	$\overset{\text{is}}{\Longleftrightarrow}$	a connected median set	$\underset{\text{induces}}{\overset{*}{\Longleftrightarrow}}$	a median graph
a set of solutions of a 2-CNF	$\overset{\text{is}}{\Longleftrightarrow}$	a median set	$\overset{\text{is}}{\Longleftrightarrow}$	$\text{fix}(f)$ of a nonexpansive f

5 Correspondence to 2-SAT

Testing satisfiability of CNF formulae (SAT) is a well-known problem in theoretical computer science; 2-SAT is a restricted variant where each clause consists of disjunction of at most two literals (i.e. propositional variables or their negations). We will consider just formulae without empty clauses. A valuation satisfying every clause of a given CNF formula φ is called a *model of* φ and the set of all models is denoted by $M(\varphi)$.

As usual, $S \models \varphi$ denotes that each valuation from S satisfies all clauses of φ. We will use the variants with a clause or formula on the left side as well; that means that every valuation which satisfies the object on the left side satisfies the right side object, as well. The last notation is for literals. For $a \in \{0,1\}$ let p_i^a denote the literal p_i if $a = 1$ and the complementary literal $\neg p_i$ if $a = 0$.

Lemma 13 *A set $S \subseteq V(Q_n)$ is a median set if and only if $S = M(\varphi)$ for some 2-CNF formula φ in n variables.*

Proof Let us start with the implication from right to left. Let S be the set of all models of a 2-CNF formula φ. To prove that S is a median set let us take three arbitrary elements (valuations) x, y, z from S and let m be their median. Further, let C be a clause from φ on variables p_i and p_j (if C is a unit clause, then $i = j$). Now we realize that the median (i.e. majority) evaluation m differs on positions i and j (i.e. the pair (m_i, m_j)) from at most two of (x_i, x_j), (y_i, y_j) and (z_i, z_j). Therefore there is at least one evaluation from x, y, z which equals to m on coordinates i and j. Since this evaluation satisfies C, we have that m satisfies C as well. Thus, we see that S is closed under medians.

For the opposite direction let S be a median set. Let us form φ by the set of all 1-clauses or 2-clauses (over the variables p_i corresponding to hypercube coordinates) that are satisfied by every point of S, i.e. $\varphi = \{C;\ S \models C,\ C \text{ is a 1-clause or a 2-clause}\}$. Clearly $S \subseteq M(\varphi)$ as it satisfies each clause of φ by definition. We need to prove that there is no other satisfying assignment in $M(\varphi)$. Let us consider $a \in M(\varphi)$. We prove that it is necessarily in S. For every $1 \le i \le j \le n$ we prove that there is some $b^{i,j}$ in S such that $b_k^{i,j} = a_k$ for all coordinates k from i to j. We will proceed by induction on $j - i$.

(a) $(i = j)$ Since a satisfies φ we have that $S \not\models \neg p_i^{a_i}$ (otherwise $\neg p_i^{a_i}$ is in φ, but then a is not a model of φ). There must be some $b^{i,i} \in S$ such that $b^{i,i} \not\models \neg p_i^{a_i}$ and that is, $b_i^{i,i} = a_i$.

(b) $(i < j)$ From induction hypothesis we have some $b^{i,j-1}$ and $b^{i+1,j}$ in S that matches a on the given parts. Since a satisfies φ, we have $S \not\models \neg p_i^{a_i} \vee \neg p_j^{a_j}$. And therefore there must be some $x \in S$ such that $x \not\models \neg p_i^{a_i} \vee \neg p_j^{a_j}$; that is $x_i = a_i$ and $x_j = a_j$. Let us define $b^{i,j} = med(b^{i,j-1}, x, b^{i+1,j})$. It is easy to see that $b^{i,j}$ equals to a on every coordinate from i to j as $b_i^{i,j-1} = x_i = a_i$, $b_k^{i,j-1} = b_k^{i+1,j} = a_k$ for every $i+1 \leq k \leq j-1$ (from induction hypothesis) and $b_j^{i+1,j} = x_j = a_j$ and $b^{i,j}$ is the median of these three vertices. It is also clear that $b^{i,j}$ is in S because S is median set.

Now for $i = 1$ and $j = n$ we get $b_k^{1,n} = a_k$ for every k, i.e. $b^{1,n} = a$ and therefore a is in S. Consequently every element of $M(\varphi)$ is also an element of S. ■

Corollary 14 *Let $S \subseteq V(Q_n)$ be a median set such that for every $i, j \in [n]$ and $a, b \in \mathbb{Z}_2$ there exist $u \in S$ with $u_i = a$ and $u_j = b$. Then $S = V(Q_n)$.*

Proof The corresponding 2-CNF φ has no clause. Otherwise, if $(x_i^a \vee x_j^b) \in \varphi$, then for every $u \in S$ we have $u_i = a$ or $u_j = b$. ■

We say that p_i is a *trivial variable of* φ if its value is the same in every satisfying assignment, in other terms its literal is a logical consequence of the formula: $\varphi \models p_i$ or $\varphi \models \neg p_i$. We will call two non-trivial variables p_i, p_j *equivalent* if $\varphi \models p_i \leftrightarrow p_j$ or $\varphi \models p_i \leftrightarrow \neg p_j$.

Lemma 15 *A set $S \subseteq V(Q_n)$ is a connected median set if and only if $S = M(\varphi)$ for some 2-CNF formula in n variables with no equivalent variables.*

Proof Let us assume that φ is a 2-CNF formula containing equivalent variables p_i and p_j, say $\varphi \models p_i \leftrightarrow p_j$ (the other case is similar), and let $S = M(\varphi)$. From nontriviality of p_i and p_j (see definition of equivalent variables) there are two valuations $a, b \in S$ with $a_i = a_j \neq b_i = b_j$. Suppose that S is connected. Then there is some $c \in S$ that agrees with a in one of the coordinates i, j and with b in the other coordinate (in order to be on the path between a and b). But that contradicts equivalence of p_i and p_j, because they would have different values in the satisfying assignment c. Thus, S is not connected.

For the other direction let $S = M(\varphi)$ be disconnected for some 2-CNF φ (S is a median set by Lemma 13). Let a and b be two points of S which are disconnected in the subgraph of hypercube induced by S such that their Hamming distance $d_H(a, b) \geq 2$ is minimal. Let i and j be some positions where a and b differ and assume that $a_i = a_j \neq b_i = b_j$ (the case $a_i \neq a_j = b_i \neq b_j$ is similar). We will show that $\varphi \models p_i \leftrightarrow p_j$. Let us assume that there is a satisfying valuation x such that $x_i = a_i$ and $x_j = b_j$ and let $m = med(a, b, x)$. From the way we have chosen x it holds that $x_i = m_i \neq m_j = x_j$. But then m is on a shortest a, b-path because S is a median set but $m \neq a, b$ which is a contradiction with the minimality of distance between a and b. Hence φ has equivalent variables. ■

6 Notes

This lecture is based on Section 3 in the thesis of Feder [3]. According to him, Lemma 13 can be found in Chung, Graham, and Saks [2], and can also be obtained from Isbell [5] and Mulder and Schrijver [6].

In the next lecture we show the remaining correspondence that a (nontrivial) set of solutions of a 2-CNF formula without equivalent variables induces a retract of Q_n. This will complete the proof that every median graph is a retract of the hypercube (Bandelt's theorem [1]). An alternative approach is described in the book of Imrich and Klavžar [4].

References

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[6] H. M. Mulder, J. Schrijver, *Median graphs and Helly hypergraphs*, Discrete Math. 25 (1979), 41–50.