Hypercube problems

Lecture 7

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1 Implication graph

Recall the notion of an *implication graph* of a 2-CNF formula φ. Its vertices are all literals and for each clause $p^a \vee q^b$ in φ we have two oriented edges (p^{1-a}, q^b) and (q^{1-b}, p^a) corresponding to implications $p^{1-a} \to q^b$ and $q^{1-b} \to p^a$. As usual, p^a denotes the literal p if $a = 1$, and the literal $\neg p$ if $a = 0$.

Definition 1 *A 2-CNF formula φ is* acyclic *(*transitively closed*) if its implication graph is acyclic (transitively closed, respectively).*

Clearly, every 2-CNF formula has an equivalent transitively closed 2-CNF formula. Furthermore, every (solvable) 2-CNF formula has an acyclic representation. (Replace all literals in a strongly connected component by a new variable.) Recall two definitions from the previous lecture. We say that p_i is a *trivial variable of* φ if $\varphi \models p_i$ or $\varphi \models \neg p_i$. Two non-trivial variables p_i and p_j are *equivalent* if $\varphi \models p_i \leftrightarrow p_j$ or $\varphi \models p_i \leftrightarrow \neg p_j$.

Observation 2 *An implication graph of a 2-CNF formula without two equivalent variables is almost acyclic, up to strongly connected components corresponding to trivial variables.*

2 2-CNF formulas and retracts (cont.)

From the previous lecture it remains to show that a (nontrivial) set of solutions of a 2-CNF formula φ without equivalent variables induces a retract of Q_n. Here, nontrivial means containing at least two elements. If φ has only a single solution $v \in V(Q_n)$, then $\{v\}$ is trivially both an image of a *weak* retraction of Q_n and a median set.

Lemma 3 *A (nontrivial) set of solutions of a 2-CNF formula φ on n variables induces a retract of Q_n.*

Proof We may assume that φ is transitively closed. First, we consider the case when φ has no trivial variables, hence it is acyclic. For a clause $C = (x_i^a \vee x_j^b)$ of φ we define a *gate*

$$g_C(x) = \begin{cases} x & \text{if } x \models C \\ x \oplus e_i \oplus e_j & \text{otherwise.} \end{cases}$$

Note that g_C is a retraction of Q_n.

Since φ is acyclic, we may order its clauses $C_1 \wedge ... \wedge C_r$ so that $C_1, ..., C_{j-1} \not\models C_j$ for every $j \leq r$. The mapping $f = g_{c_1} g_{c_2} ... g_{c_r}$ (a boolean circuit) is an endomorphism on Q_n. For $S = M(\varphi)$ clearly

$$u \in S \Rightarrow g_{c_i}(u) = u \text{ for every } i \Rightarrow f(u) = u;$$

that is, every solution of φ is a fixed point of f. Thus, it suffices to show that $rng(f) \subseteq S$.

Claim 4 *For every $l \geq 0$, if $u \in rng(g_{C_1} g_{C_2} ... g_{C_l})$ then $u \models C_1 \wedge ... \wedge C_l$.*

Proof We proceed by induction on l. For $l = 0$ the claim is vacuous. Let $C_l = (x_i^a \vee x_j^b)$, $v \in rng(g_{C_1} g_{C_2} \ldots g_{C_{l-1}})$ and $u = g_{C_l}(v)$, so $u \models C_l$. We have to show that $u \models C_p$ for every $p < l$. We distinguish the following cases.

1. C_p does not contain x_i nor x_j. Then apply induction for $v = u$.
2. $C_p = (x_i^a \vee x_k^c)$ for some $k \neq j$ and $c \in \{0, 1\}$. Then $v_i = a \Rightarrow u_i = a$ or $v_l = c \Rightarrow u_k = v_l = c$.
3. $C_p = (\neg x_i^a \vee x_k^c)$ for some $k \neq j$ and $c \in \{0, 1\}$. Then $C_p, C_l \models x_j^b \vee x_k^c$. Since φ is transitive and by our order of clauses, this clause is already satisfied by v (by induction). Then $v_j = b \Rightarrow u_j = v_j = b$ or $v_k = c \Rightarrow u_k = v_k = c$.
4. $C_p = (x_i^a \vee \neg x_j^b)$. Then $C_p, C_l \models x_i^a$ and thus x_i is a trivial variable. So this case does not occur by the assumption that φ has no trivial variables. Similarly for $C_p = (\neg x_i^a \vee x_j^b)$.
5. $C_p = (\neg x_i^a \vee \neg x_j^b)$. Then $C_p, C_l \models x_i^a \leftrightarrow x_j^b$. Thus x_i and x_j are equivalent variables. Hence this case does not occur by the assumption that φ has no equivalent variables.

■

Applying Claim 4 for $l = r$, we obtain that $u \in rng(f)$ implies $u \in S$. Therefore, $rng(f) = fix(f) = S$ and f is the desired retraction on Q_n.

If φ has a trivial variable x_i^a, then we eliminate it from φ and we apply an endomorphism

$$h_{ij}(x) = x \oplus e_i \oplus e_j \text{ if } x_i \neq a, \text{ and } h_{ij}(x) = x \text{ otherwise,}$$

where j is an index of some nontrivial variable. Such variable exists since S is nontrivial. ■

By Lemma 3 and results in the previous lecture, we conclude as follows.

Theorem 5 *Let $S \subseteq V(Q_n)$ be a nontrivial set. The following statements are equivalent.*

- *S is a connected median set,*
- *S induces a median graph,*
- *S is a set of solutions of a 2-CNF formula with no two equivalent variables,*
- *$S = rng(f)$ for some retraction f on Q_n (i.e. S induces a retract of Q_n).*

A similar result holds for (not connected) median sets. We mention it without proof.

Theorem 6 *A set $S \subseteq V(Q_n)$ is a median set if and only if $S = fix(f)$ for some nonexpansive map f on Q_n if and only if S is a set of solutions of some 2-CNF formula.*

3 Fixed cube theorem

Definition 7 *The distance center of a connected graph $G = (V, E)$ is a set of vertices x that minimize the sum $\sum_{y \in V} d(x, y)$.*

Lemma 8 *The distance center of a connected median set $S \subseteq V(Q_n)$ is a subcube of Q_n.*

Proof Let $S = rng(f)$ of a corresponding retraction f. Let C be the set of vertices x of Q_n that minimize $\sum_{y \in S} d(x, y)$. For every $x \in V(Q_n)$,

$$\sum_{y \in S} d(f(x), y) = \sum_{y \in S} d(f(x), f(y)) \leq \sum_{y \in S} d(x, y).$$

If $x \notin S$, then $d(f(x), f(x)) = 0 < d(x, f(x))$, so $x \notin C$. Therefore $C \subseteq S$. Since we may rewrite the sum $\sum_{y \in S} d(x, y) = \sum_{i \in [n]} \sum_{y \in S} d(x_i, y_i)$ by coordinates, let us define $s \in \{0, 1, *\}^n$ by

$$s_i = \begin{cases} x_i & \text{if } \sum_{y \in S} d(x_i, y_i) < \sum_{y \in S} d(\overline{x_i}, y_i), \\ * & \text{if } \sum_{y \in S} d(x_i, y_i) = \sum_{y \in S} d(\overline{x_i}, y_i). \end{cases}$$

Then $C = Q_n[s]$, which is a subcube of Q_n. ■

Definition 9 *A* fixed cube *C of a nonexpansive map f on Q_n is a subcube such that $f(C) = C$ (not necessarily point-wise.)*

Theorem 10 *Every nonexpansive map f on Q_n has a fixed cube.*

Proof Recall that $S = per(f)$ is a connected median set, $f \upharpoonright S$ is an automorphism of $\langle S \rangle$. The distance center C is invariant under automorphisms. Thus $f \upharpoonright C$ is an automorphism of $\langle C \rangle$. By Lemma 8, it is a subcube and hence a fixed cube of f. ■

Corollary 11 *Every median graph contains a subcube that is invariant under all automorphisms (nonexpansive maps).*

Problem 1 (Feder [1]) *Find a necessary and sufficient condition on four pairs (x_i, y_i) of vertices to have a nonexpansive map f with $f(x_i) = y_i$.*

Note that for three pairs it is both necessary and sufficient that $d(y_i, y_j) \leq d(x_i, x_j)$ for all i, j.

Problem 2 *Find new classes of subgraphs of Q_n with interesting correspondences.*

4 Notes

The references for this lecture are the same as for the previous lecture. The main reference is the thesis of Feder [1] with an alternative approach presented in the book of Imrich and Klavžar [2].

References

[1] T. Feder, *Stable networks and product graphs*, Mem. Amer. Math. Soc. 116 No.555 (1995), 223 pp.

[2] W. Imrich, S. Klavžar, *Product graphs, structure and recognition*, Wiley, New York, 2000.