Hypercube problems

Lecture 8

November 28, 2012

Lecturer: Petr Gregor Scribe by: Kryštof Měkuta, Markéta Calábková Updated: February 5, 2020

1 Matchings - direction characteristics

A *direction characteristics* (a.k.a a projection vector, a spectrum) of a matching M in Q_n is $\chi(M) = (a_1, a_2, \dots, a_n)$ where a_i is the number of the edges of M in the direction i.

A sequence $(a_1, a_2, \dots, a_n)$ of nonnegative integers is called *implementable* if $\chi(M) = (a_1, a_2, \dots, a_n)$ for some matching M in Q_n. A sequence $(a_1, a_2, \dots, a_n)$ is clearly not implementable if $\sum_{i\in[n]} a_i > 2^{n-1}$.

Theorem 1 *A sequence $(a_1, \dots, a_n)$ with $\sum_{i\in[n]} a_i = 2^{n-1}$ and $n \geq 2$ is implementable (by a perfect matching) if and only if a_i is even for every $i \in [n]$.*

Proof If a_i is odd for some $i \in [n]$ and there is a perfect matching M implementing $(a_1, \dots, a_n)$, then both $Q_n^{i,0}$, $Q_n^{i,1}$ have vertices that are unmatched by M. Then M is not a perfect matching, a contradiction with the sum 2^{n-1} of edges of M.

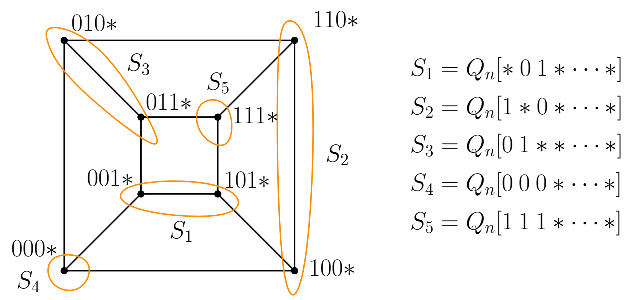

Figure 1: A partition of Q_n into subcubes $S_1, \dots, S_5$.

On the other hand, for $n = 2, 3, 4$ the statement holds by inspection of all cases. Assume that $a_1 \leq a_2 \leq \cdots \leq a_n$. We partition Q_n into five subcubes $S_1, \dots, S_5$ as depicted on Figure 1. Furthermore, for every $4 \leq i \leq n$ we partition $a_i = a_i^1 + \cdots + a_i^5$ so that

$$a_k + \sum_{i=4}^{n} a_i^k = 2^{n-3} \quad \text{for } k = 1,2,3, \text{ and}$$

$$\sum_{i=4}^{n} a_i^k = 2^{n-4} \quad \text{for } k = 4,5.$$

Such partitions exist since $a_1, a_2, a_3 \leq 2^{n-3}$, otherwise $\sum_{i=3}^{n} a_i \geq (n-2)(2^{n-3}+2) > 2^{n-1}$ for $n \geq 5$. Finally, we apply induction for $S_1, \dots, S_5$. ■

Perhaps surprisingly, for non-perfect matchings, there is no restriction on a_i's.

Theorem 2 *Every sequence* $(a_1, a_2, \ldots, a_n)$ *with* $\sum_{i\in[n]} a_i < 2^{n-1}$ *is implementable.*

The proof uses similar partitions. Without loss of generality we may assume $\sum a_i = 2^{n-1}-1$ for induction, and use automorphisms on subcubes to turn them so that we can add one edge into direction where needed. The proof is a bit technical and we omit it here.

Problem 1 *Extend the characterizations of implementable sequences to higher dimensions, for example for families of vertex-disjoint* 2*-dimensional subcubes instead of edges.*

2 Maximal matchings

A matching M in G is *maximal* if all neighbors of every unmatched vertex are matched. Let $m(G)$ be the smallest size of a maximal matching of G. In this section we determine the asymptotics of $m(Q_n)$.

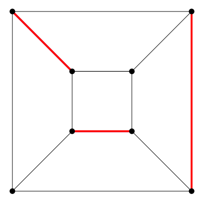

Figure 2: A (non-perfect) maximal matching in Q_3.

Observation 3 *In every maximal matching of* Q_n *at least one third of the remaining edges have both vertices matched.*

Proof Let M be a maximal matching of Q_n and $R = E(Q_n) \setminus M$. To every edge $uv \in R$

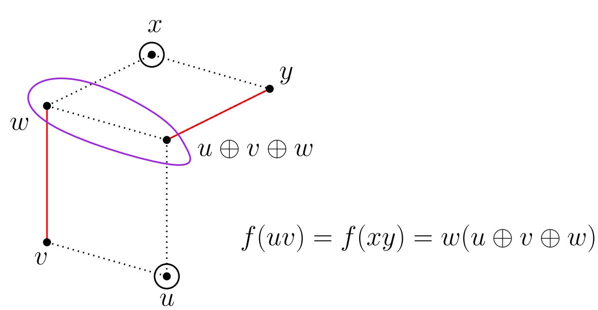

Figure 3: The assignment of edge of mapping f

we assign an edge $f(uv) \in R$ with both vertices matched as follows. If both u, v are matched, then $f(uv) = uv$. Otherwise, assuming that v is matched by $vw \in M$, we put $f(uv) = w(u \oplus v \oplus w)$, see Figure 3. Then every edge of R with both vertices matched has at most three edges of R assigned to it. ■

Corollary 4

$$m(Q_n) \geq \frac{n}{3n-1} 2^n \textit{ for every } n \geq 1.$$

Proof Let M be a maximal matching of Q_n, $m = |M|$, m_1 be the number of edges with one matched vertex and m_2 be the number of edges with both vertices matched. From Observation 3 we have

$$0 \leq 2m_2 - m_1 = 3(2m_2 + m_1) - 4(m_2 + m_1). \quad (1)$$

By counting incidences between matched vertices and edges that are not in the matching, we have

$$2m_2 + m_1 = 2(n-1)m. \quad (2)$$

Since $n2^{n-1} = E(Q_n) = m + m_1 + m_2$, it is clear that

$$m_1 + m_2 = n2^{n-1} - m. \quad (3)$$

By substituting (2) and (3) to (1), we obtain the statement. ■

Lemma 5

$$m(Q_{n+1}) \leq 2m(Q_n) \textit{ for every } n \geq 1.$$

Proof We split Q_{n+1} along the direction $n+1$ into subcubes Q_{n+1}^0 and Q_{n+1}^1. Let M_0 be a maximal matching in Q_{n+1}^0. Then, $M_1 = M_0 \oplus e_1 \oplus e_{n+1}$ is a maximal matching in Q_{n+1}^1.

Every vertex v in Q_{n+1}^0 that is unmatched by M_0 has its neighbor $x = v \oplus e_1$ matched by M_0 since M_0 is maximal. But then $v \oplus e_{n+1} = x \oplus e_1 \oplus e_{n+1}$ is matched by M_1. Therefore, the matching $M_0 \cup M_1$ is a maximal matching in Q_{n+1} of size $2|M_0|$. ■

By Lemma 5, the sequence $m(Q_n)/2^n$ is non-increasing, and therefore has a limit. By Corollary 4, this limit is at least 1/3.

Theorem 6

$$\lim_{n\to\infty} \frac{m(Q_n)}{2^n} = \frac{1}{3}$$

Proof We may assume that $6 \mid n$. Let $L_0 \cup L_1 \cup \ldots \cup L_n$ be a partition of $V(Q_n)$ into levels, and for $k = 0, 1, 2$ let

$$C_k = \bigcup_{\substack{i \equiv k \bmod 3 \\ i \leq n/2}} (L_i \cup L_{n-i}).$$

If $S \subseteq L_i$ with $i < n/2$, then $|N(S) \cap L_{i+1}| \geq |S|\frac{n-i}{i+1}$. Therefore, by Hall's theorem, there is a matching $M_1 \subseteq E(C_1, C_2)$ that is maximal in $C_1 \cup C_2$; i.e. it covers C_1. Similarly, there is $M_2 \subseteq E(C_2, C_0)$ covering the remaining vertices of C_2 (except possibly few vertices in levels $n/2-1$, $n/2+1$). Observe that the matching $M_1 \cup M_2$ is maximal. Furthermore,

$$|M_1 \cup M_2| \leq 2 \sum_{\substack{i \leq n/2 \\ i \equiv 2 \bmod 3}} \binom{n}{i} = m,$$

and it can be shown that $m = 2^n(\frac{1}{3} + o(1))$, which is left as an exercise. ■

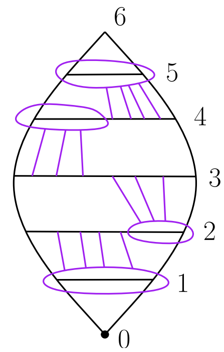

Figure 4: A maximal matching of Q_6 constructed in Theorem 6.

3 Forcing number

Let $M \subseteq E(G)$ be a perfect matching of a graph G. A set $S \subseteq M$ *forces* M if S is in no other perfect matching of G. The forcing number $f(M)$ of M is the minimal size of a forcing subset $S \subseteq M$. Let $f_{\min}(G)$ be the minimal forcing number over all perfect matchings of G.

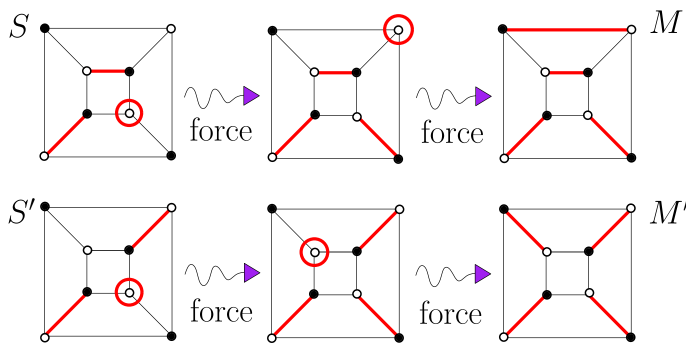

Figure 5: Forcing of two perfect matchings of Q_3.

Observation 7 *For each set $S \subseteq E(G)$ that forces a perfect matching the induced subgraph $G \setminus V(S)$ has a unique perfect matching.*

How many vertices can be in an induced subgraph of G with a unique perfect matching? From an upper bound on this number we can get a lower bound on $|S|$, therefore we bound $f_{\min}(G)$.

4 Lower bound on $f_{\min}(G)$

From now, let $G = (U \cup V, E)$ be an undirected bipartite graph with partitions $U = \{u_1, \ldots, u_{|U|}\}$ and $V = \{v_1, \ldots, v_{|V|}\}$. The *biadjacency matrix* of G is a matrix $B = (b_{ij})^{|U| \times |V|}$ over $\mathbb{N}$ in which $b_{ij} = 1$ if $u_i v_j \in E$ and $b_{ij} = 0$ elsewhere. Let $\mathbb{F}$ be an arbitrary field. Let us arbitrarily assign nonzero weights $w_{ij} \in \mathbb{F}$ to edges $u_i v_j \in E$. A *weighted* *biadjacency matrix* of G is a matrix $W = (w'_{ij})^{|U|\times|V|}$ where $w'_{ij} = w_{ij}$ if $u_i v_j \in E$ and $w'_{ij} = 0$ elsewhere.

Lemma 8 *Let H be an induced subgraph of a bipartite graph G with a unique perfect matching, let W be a weighted biadjacency matrix of G with values from a field $\mathbb{F}$. Then $|H| \leq 2\,\mathrm{rank}(W)$, where $|H|$ denotes the number of vertices in H.*

Proof Since H has a perfect matching, both its partitions have the same size. Let B_H be the submatrix of W induced by H. Then W_H is a square matrix of size $|H|/2$ and has a nonzero determinant because H has an unique perfect matching, so its rank is $|H|/2$. Then

$$|H| = 2\,\mathrm{rank}(W_H) \leq 2\,\mathrm{rank}(W)\,.$$

■

Recall that Q_n has a biadjacency matrix M_n that looks as follows:

$$M_1 = (1),\ M_n = \begin{pmatrix} M_{n-1} & I \\ I & M_{n-1} \end{pmatrix} \tag{4}$$

It holds that M_n is regular if and only if n is odd. We will need regularity, so we have to look for a regular alternative.

Lemma 9 *For every $n \geq 1$ there is a regular matrix A_n over $\mathbb{Z}_3$ such that both A_n and A_n^{-1} are weighted biadjacency matrices of Q_n.*

Proof First, let $A_1 = A_1^{-1} = (1)$. For $n \geq 1$ we assume A_n is regular and construct A_{n+1} recursively as

$$A_{n+1} = \begin{pmatrix} 2A_n & I \\ I & A_n^{-1} \end{pmatrix}. \tag{5}$$

Note that in $\mathbb{Z}_3$, $2+2 = 2\cdot 2 = 1$ and $1+2 = 0$. It can be easily verified that

$$A_{n+1}^{-1} = \begin{pmatrix} A_n^{-1} & 2I \\ 2I & 2A_n \end{pmatrix}. \tag{6}$$

Hence A_{n+1} is also regular, because it has an inverse. By comparing the recursive constructions (5) and (6) with the equations in (4) we can see that both A_n and A_n^{-1} are weighted biadjacency matrices of Q_n. ■

Theorem 10 *Any induced subgraph H of Q_n, $n \geq 2$, with a unique perfect matching contains at most 2^{n-1} vertices.*

Proof Let A_{n-1} be the matrix as in (5). Note that A_{n-1} is regular, so it has rank 2^{n-2}. Let

$$B_n = \begin{pmatrix} A_{n-1} & I \\ I & A_{n-1}^{-1} \end{pmatrix}. \tag{7}$$

Then $2^{n-2} \leq \operatorname{rank}(B_n) \leq 2^{n-1}$. Moreover, for all vectors $\mathbf{x} \in \mathbb{Z}_3^{2^{n-2}}$ we get

$$B_n \begin{pmatrix} \mathbf{x} \\ 2A_{n-1}\mathbf{x} \end{pmatrix} = 0\,, \tag{8}$$

so B_n has at most 2^{n-2} linearly independent columns. We get $\operatorname{rank}(B_n) = 2^{n-2}$ and from Lemma 8 it follows $|H| \leq 2^{n-1}$. ■

Corollary 11 *For every $n \geq 2$, $f_{\min}(Q_n) = 2^{n-2}$.*

Proof Let S be the minimal forcing set of a perfect matching M on Q_n. Then by Observation 7 and Theorem 10 we get $|Q_n \setminus V(S)| \leq 2^{n-1}$, so $|S| \geq 2^{n-2}$. On the other hand, the perfect matching $M = \{uv \mid u \oplus v = e_1\}$ is forced by a set $S = \{uv \in M \mid u_1 = 0 \text{ and } |u| \text{ even}\}$, which has size 2^{n-2}. ■

5 Semi-induced matchings

A matching M of G is *induced* if it is induced by its vertices (it is an induced subgraph of G), i.e. no two edges of M are adjacent. In other words, both vertices of each edge in M have no neighbour covered by M. A matching M of G is *semi-induced* if every edge of M has at least one vertex which has no neighbour covered by M.

Corollary 12 *The maximal size of a semi-induced matching in Q_n, $n \geq 2$, is 2^{n-2}.*

Proof If M is a semi-induced matching, then the subgraph induced by $V(M)$ has a unique perfect matching, so $|V(M)| \leq 2^{n-1}$ and $|M| \leq 2^{n-2}$ by Theorem 10. On the other hand, in the proof of Corollary 11 we have found an induced (which is also semi-induced) matching of size 2^{n-2}. ■

Note

Section 1 is based on the work of Felzenbaum, Holzman and Kleitman [2]. Section 2 is based on the work of Forcade [3]. His conjecture that $m(Q_n) = \lceil \frac{n2^n}{3n-1} \rceil$ was disproved by Havel and Křivánek [5] by showing that $m(Q_6) \geq 24$. The exact value of $m(Q_n)$ is known only for small values of n. Sections 3 and 4 are based on the work of [1]. Section 4 solves Conjecture 1 in [4].

References

[1] A. A. Divan, *The minimum forcing number of perfect matchings in the hypercube*, arXiv, 1712.03535 (2018).

[2] A. Felzenbaum, R. Holzman, D. J. Kleitman, *Packing lines in a hypercube*, Discrete Mathematics 117 (1993), 107–112.

[3] R. Forcade, *Smallest maximal matchings in the graph of the d-dimensional cube*, Journal of Combinatorial Theory B 14 (1973), 153–156.

[4] A. Gyárfas, A. Hubenko, *Semistrong edge coloring of graphs*, J. Graph Theory 49 (2005), 39–47.

[5] I. Havel, M. Křivánek, *On maximal matchings in Q6 and a conjecture of R. Forcade*, Comment. Math. Univ. Carolin. 23 (1982), 123–136.

[6] L. Pachter, P. Kim, *Forcing matchings on square grids*, Discrete Mathematics 190 (1998), 287–294.

[7] M. E. Riddle, *The minimum forcing number for the torus and hypercube*, Discrete Mathematics 245 (2002), 283–292.