**Hypercube problems**

Lecture 9

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# 1 The number of perfect matchings

We determine asymptotically the number of perfect matchings of $Q_n$. Let us recall that in a balanced bipartite graph $G$ on $2m$ vertices the number of perfect matchings $\mathrm{pm}(G)$ is the *permanent* $\mathrm{per}(M_G)$ of its biadjacency $m \times m$ matrix $M_G = (m_{ij})$; that is,

$$\mathrm{pm}(G) = \mathrm{per}(M_G) = \sum_{\pi \in S_m} \prod_{i=1}^{m} m_{i,\pi(i)}.$$

A real matrix is *doubly stochastic* if all its elements are nonnegative and every row and every column sums to 1. The following result says that the minimal permanent over all doubly stochastic matrices is attained by the uniform matrix; that is, the matrix with $1/m$ in each position. It was posed as a conjecture by van der Waerden[1] [24] and it took half a century till it was proved, independently, by Egoryčev [5] and Falikman [6].

**Theorem 1 (van der Waerden's conjecture [5, 6])** *For every doubly stochastic $m \times m$ matrix $M$,*

$$\mathrm{per}(M) \geq \frac{m!}{m^m}.$$

There is also an upper bound on permanent of binary matrices based on row sums that was conjectured by Minc [16].

**Theorem 2 (Brègman [3])** *For every binary $m \times m$ matrix $M$ with sum $r_i$ in the $i$-th row for $1 \leq i \leq m$,*

$$\mathrm{per}(M) \leq \prod_{i=1}^{m} (r_i!)^{1/r_i}.$$

**Corollary 3** *For every $n$-regular balanced bipartite graph $G$ on $2m$ vertices,*

$$n^m \frac{m!}{m^m} \leq \mathrm{pm}(G) \leq (n!)^{\frac{m}{n}}.$$

Using a standard approximation of $m!$ we get

$$\mathrm{pm}(G) \geq \left(\frac{n}{m}\right)^m \left(\frac{m}{e}\right)^m = \left(\frac{n}{e}\right)^m. \tag{1}$$

[1] There is another result named after him, the van der Waerden's theorem on arithmetic progressions.

In fact, for the hypercube it can be computed [4] that

$$\mathrm{pm}(Q_n) = \left[\frac{n}{e}(1 + o(1))\right]^{2^{n-1}}.$$

The exact value of $\mathrm{pm}(Q_n)$ is known for $n \leq 7$ [20].

## 2 Maximal forcing number

Recall from the previous lecture that the forcing number $f(M)$ of a perfect matching $M$ of a graph $G$ is the minimal size of a subset $S \subseteq M$ that is not in other perfect matching of $G$. The maximal forcing number $f_{max}(G)$ of $G$ is the maximal $f(M)$ over all perfect matchings $M$ of $G$.

Perhaps surprisingly, compared to the results on minimal forcing number, the following proposition shows that $f_{max}(Q_n)$ grows above any constant fraction of a perfect matching.

**Proposition 4 (Alon, see [18])** *For any constant $0 < c < 1$ there exists a perfect matching $M$ in $Q_n$ for sufficiently large $n$ with $f(M) > c2^{n-1}$.*

**Proof** We proceed by contradiction. If $f_{\max}(Q_n) \leq c2^{n-1}$ then there are at least $pm(Q_n)$ forcing sets of size at most $c2^{n-1}$. The number of such matchings is at most $2^{2^{n-1}} n^{c2^{n-1}}$ (the number of subsets of one bipartite class multiplied by the number of ways to choose directions for edges from the selected vertices). Thus, applying (1), we have

$$\left(\frac{n}{e}\right)^{2^{n-1}} < \mathrm{pm}(Q_n) < 2^{2^{n-1}} n^{c2^{n-1}}.$$

This gives us the inequality

$$\frac{n}{e} < 2n^c$$

which is false if $n$ is large. ■

## 3 Fink's theorem

Kreweras [14] conjectured that every perfect matching in $Q_n$ for $n \geq 2$ extends to a Hamiltonian cycle. Fink [7] confirmed this assertion by proving a more general result[2]. Let $K(Q_n)$ denote the complete graph on $V(Q_n) = \mathbb{Z}_2^n$.

**Theorem 5 (Fink [7])** *For every perfect matching $P$ of $K(Q_n)$ where $n \geq 2$ there is a perfect matching $R$ of $Q_n$ such that $P \cup R$ forms a Hamiltonian cycle in $K(Q_n)$.*

**Proof** We proceed by induction on $n$. For $n = 2$ there are three cases as shown on Figure 1. Assuming the statement holds for $n - 1$, we prove it for $n$. We divide $Q_n$ into $Q_{n-1}^{i,0}$ and $Q_{n-1}^{i,1}$ along some direction $i$ that is crossed by some edge of $P$. Let $P'$ denote the set of edges from $P$ crossing the direction $i$, and let $P_0$ and $P_1$ denote the rest of edges from $P$ in $Q_{n-1}^{i,0}$ or $Q_{n-1}^{i,1}$, respectively. Note that $|P'|$ is even.

[2]His proof is a textbook example that considering a stronger statement often helps in induction.

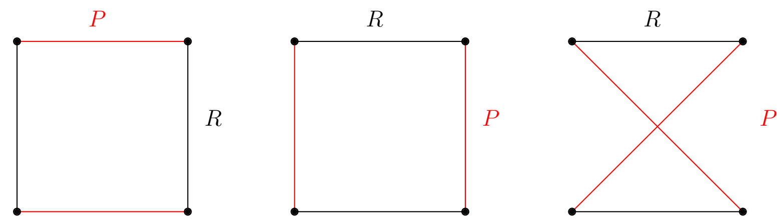


**Figure 1**: Three cases for $n = 2$.

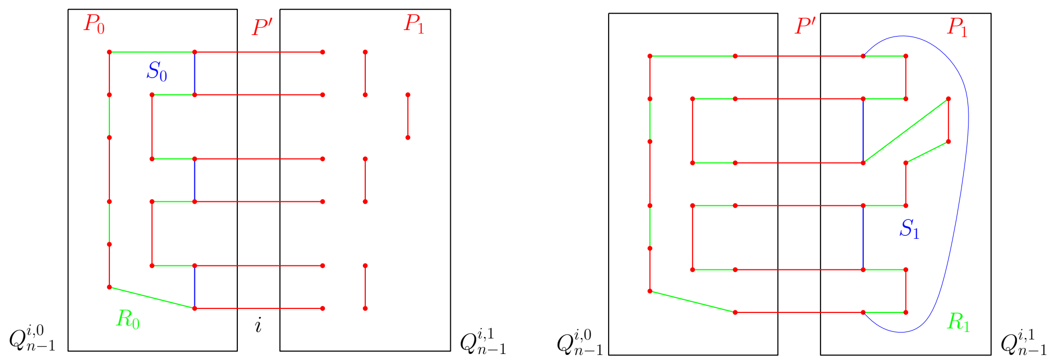


**Figure 2**: (a) Hamiltonian cycle in $Q^{i,0}_{n-1}$, (b) Hamiltonian cycle in $Q^{i,1}_{n-1}$.

First, we focus on $Q^{i,0}_{n-1}$, see Figure 2(a). Choose an arbitrary perfect matching $S_0$ (blue) on the endvertices of $P'$ in $Q^{i,0}_{n-1}$. Then by the induction hypothesis for $P_0 \cup S_0$ we obtain a perfect matching $R_0$ of $Q^{i,0}_{n-1}$ (green) such that $P_0 \cup S_0 \cup R_0$ forms a Hamiltonian cycle in $K(Q^{i,0}_{n-1})$.

Then, we "delete" the edges $S_0$ and focus on $Q^{i,1}_{n-1}$, see Figure 2(b). Let $S_1$ (blue) be the perfect matching on the endvertices of $P'$ in $Q^{i,1}_{n-1}$ such that every edge from $S_1$ represents the path between the same vertices formed by $P' \cup P_0 \cup R_0$. Applying induction for $P_1 \cup S_1$ we obtain a perfect matching $R_1$ of $Q^{i,1}_{n-1}$ (green) such that $P_1 \cup S_1 \cup R_1$ forms a Hamiltonian cycle in $K(Q^{i,1}_{n-1})$. By "deleting" the edges $S_1$ we have a desired matching $R = R_0 \cup R_1$. ■

In case that $P$ is a perfect matching of $Q_n$, Fink's theorem can be strengthened to allow forbidden edges. For other related results, see his thesis [9].

**Theorem 6 (Fon-der-Flaass [10])** *Let $F \subseteq E(Q_n)$ for $n \geq 4$ be a set such that every subcube of dimension 4 contains at most one edge of $F$. Then every perfect matching of $Q_n - F$ extends to a Hamiltonian cycle in $Q_n - F$.*

For $n \geq 2$ let $M_n$ denote the graph whose vertices are all perfect matchings of $Q_n$ and edges join two perfect matchings that form a Hamiltonian cycle of $Q_n$. Fink proved also the second Kreweras conjecture [14].

**Theorem 7 (Fink [8])** *For every $n \geq 4$ the graph $M_n$ is bipartite and connected.*

In case of general (non-perfect) matchings the question remains open.

**Problem 1 (Ruskey, Savage [19])** *Does every matching of $Q_n$ for $n \geq 2$ extend to a Hamiltonian cycle?*

## 4 Extending to $f$-factors

An *$f$-factor* of a graph $G$ for a function $f : V(G) \to \mathbb{N}$ is a spanning subgraph $H$ of $G$ such that $\deg_H(v) = f(v)$ for every $v \in V(G)$. For a set $S \subseteq V(G)$ let $f(S) = \sum_{v \in S} f(v)$. The following theorem is a useful extension of Hall's matching theorem.

**Theorem 8 (Ore-Ryser [17])** *A bipartite graph $G = (A \cup B, E)$ has an $f$-factor if and only if $f(A) = f(B)$ and for every $S \subseteq A$,*

$$f(S) \leq \sum_{b \in N(S)} \min(f(b), |N_S(b)|).$$

Applying this theorem it was shown [23] that every matching in $Q_5$ extends to a 2-factor (that is, a cover of $Q_5$ by vertex-disjoint cycles). This is also true for $n = 2, 3, 4$. Problem 1 can be relaxed as follows.

**Problem 2 (Vandenbussche, West [23])** *Does every matching of $Q_n$ for $n \geq 2$ extend to a 2-factor?*

## 5 Hamiltonian decomposition

A *Hamiltonian decomposition* of an $2n$-regular graph $G$ is a partition of $E(G)$ into $n$ Hamiltonian cycles of $G$.

For example, $Q_4$ has a Hamiltonian decomposition shown on Figure 3 with the red and blue Hamiltonian cycles. Observe that the red cycle is obtained from the blue one by shifting it two steps down and flipping it horizontally.

We will see that $Q_{2n}$ has a Hamiltonian decomposition for every $n \geq 1$. First, we introduce an auxiliary definition and we state a key lemma. A matching $M$ is *orthogonal* to a Hamiltonian decomposition $C_1, \ldots, C_n$ if $|M \cap C_i| = 1$ for every $i \in [n]$.

**Lemma 9 (Stong [21])** *Let $G = (V, E)$ be a bipartite graph with a matching $M$ orthogonal to a Hamiltonian decomposition $C_1, \ldots, C_n$ of $G$. Then $G \,\square\, C_4$ has a Hamiltonian decomposition as well.*

**Proof** Let $m = |V|$ and assume that $C_i = (c_i^1, \ldots, c_i^m)$ with $c_i^m c_i^1 \in M$ for all $i \in [n]$. For $1 \leq j \leq n - 1$ we create a new Hamiltonian cycle $C'_j$ of $G \,\square\, C_4$ by joining four copies of $C_j$ in alternating order; that is,

$$C'_j = ((C_j, 1), (C_j^R, 2), (C_j, 3), (C_j^R, 4))$$

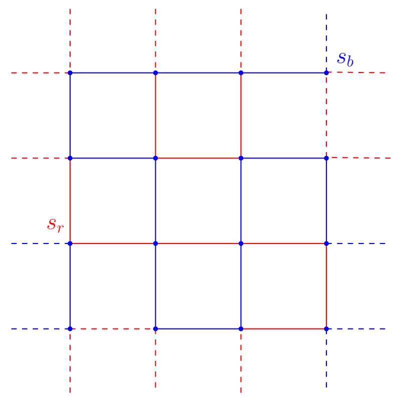


**Figure 3**: A partition of $Q_4$ into two Hamiltonian cycles (drawn on a torus).

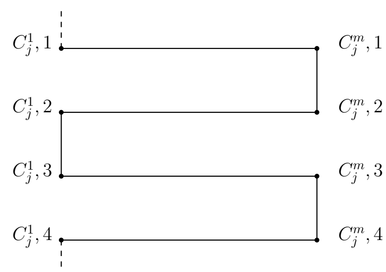


**Figure 4**: The cycle $C'_j$ from four copies of $C_j$ for $1 \leq j \leq n-1$.

as is shown in Figure 4. Since $c_{j_1}^m c_{j_1}^1$ and $c_{j_2}^m c_{j_2}^1$ are disjoint and $C_{j_1}$, $C_{j_2}$ are edge-disjoint in $G$, it follows that every two cycles $C'_{j_1}$ and $C'_{j_2}$ are edge-disjoint for distinct $1 \leq j_1, j_2 \leq n-1$.

It remains to decompose $(G \,\square\, C_4) - \bigcup_{j=1}^{n-1} C'_j$ into two Hamiltonian cycles $C'_n$ and $C'_{n+1}$. For this purpose we will use the copies of the cycle $C_n$ in $G$. We start with the Hamiltonian cycles $\overline{C}_n$, $\overline{C}_{n+1}$ that decompose $C_n \,\square\, C_4$ as shown on Figure 5.

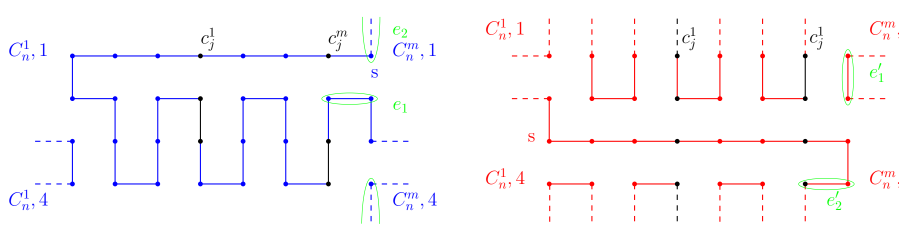


**Figure 5**: The cycles $\overline{C}_n$ and $\overline{C}_{n+1}$.

However, we are not finished yet. For every $j = 1, \ldots, n-1$ we have to correct the cycles $\overline{C}_n$, $\overline{C}_{n+1}$ as on Figure 6 and we obtain the desired cycles $C'_n$, $C'_{n+1}$. Instead of the edges $(c_j^m, 1)(c_j^m, 2)$, $(c_j^1, 2)(c_j^1, 3)$, $(c_j^m, 3)(c_j^m, 4)$, $(c_j^1, 4)(c_j^1, 1)$ that are used already by $C'_j$, we use the edges $(c_j^1, k)(c_j^m, k)$ for $k = 1, \ldots, 4$. ■

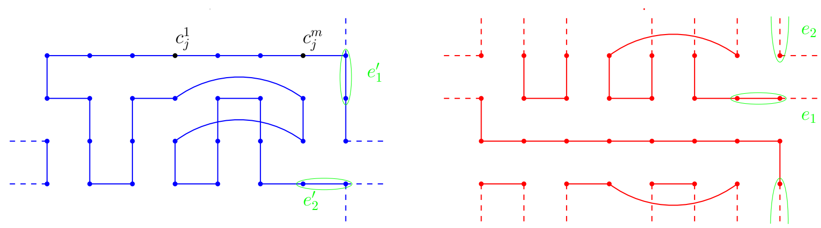


**Figure 6**: The cycles $C_n'$ and $C_{n+1}'$.

The above lemma can be directly extended for $C_{2k}$ instead of $C_4$, non-bipartite graphs, or (less directly) for $C_{2k+1}$ instead of $C_{2k}$ [21].

**Corollary 10** *$Q_{2n}$ has a Hamiltonian decomposition for every $n \geq 1$.*

**Proof** We have $Q_{2n} \simeq C_4^n$ (the $n$th Cartesian power). The statement follows by Lemma 9 by induction with the trivial base for $Q_2$. Since $4(n-1) < 2^{2n}$ for every $n \geq 1$, an orthogonal matching exist in each induction step. Apply Lemma 9 by induction. ■

Note that for $n = 2$ we obtain the decomposition of $Q_4$ on Figure 3.

Obviously, $Q_n$ nas no Hamiltonian decomposition when $n$ is odd. However, we can find some when we switch to directed cubes. Let $\overleftrightarrow{Q_n}$ be the directed graph obtained from $Q_n$ by replacing each edge with two directed edges of opposite directions.

**Theorem 11 (Stong [22])** *$\overleftrightarrow{Q_n}$ for any $n \neq 3$ has a decomposition into $n$ directed Hamiltonian cycles.*

The proof is similar to the proof of Lemma 9.

## 6 Semi-perfect 1-factorization

A 1-*factorization* of a (regular) graph $G = (V, E)$ is a partition of $E$ into 1-factors (perfect matchings). A 1-factorization $F_1, \ldots, F_n$ is *perfect* if $F_i \cup F_j$ induces a Hamiltonian cycle for every distinct $1 \leq i, j \leq n$. A long-standing conjecture on perfect 1-factorization of complete graphs is as follows.

**Conjecture 12 (Kotzig [12])** *$K_{2n}$ has a perfect 1-factorization for every $n \geq 2$.*

Let us now state an interesting corollary of Theorem 11.

**Corollary 13** *For $n \neq 3$ there are two 1-factorizations $\mathcal{A} = A_1, \ldots, A_n$ and $\mathcal{B} = B_1, \ldots, B_n$ of $Q_n$ such that $A_i \cup B_i$ is a Hamiltonian cycle of $Q_n$ for every $1 \leq i \leq n$.*

**Proof** Let $H_1, \ldots, H_n$ be a HC-decomposition of $\overleftrightarrow{Q_n}$ and let $A_i, B_i$ be a decomposition of $H_i$ where $A_i$ contains the edges from even to odd vertices and $B_i$ contains the rest. Let us now drop the orientation of the edges in $\mathcal{A}$ and $\mathcal{B}$. Every edge of $Q_i$ is now contained exactly once in $\mathcal{A}$ and exactly once in $\mathcal{B}$, so $\mathcal{A}$ and $\mathcal{B}$ are correct 1-factorizations of $Q_i$. ■

A 1-factorization $F_1, \ldots, F_n$ is *k-semi-perfect* if $F_i \cup F_j$ forms a Hamiltonian cycle for every $1 \leq i \leq k < j \leq n$.

**Theorem 14 (Behague [2])** *For every $k \neq 3$ and $l \neq 3$ there is a k-semi-perfect factorization of $Q_{k+l}$.*

**Proof** Let $Q_{k+l} = Q_k \,\square\, Q_l$ and let $A_1, \ldots, A_k$ and $B_1, \ldots, B_k$ be 1-factorizations for $Q_k$ and $X_1, \ldots, X_l$ and $Y_1 \ldots, Y_l$ be 1-factorizations for $Q_l$ as in Corollary 13. Let $A_i^v, B_i^v$ denote the copies of $A_i, B_i$ in the $v^{\text{th}}$ fiber of $Q_k$ (in the subcube $*^k v$, where $v \in \{0,1\}^l$). Similarly $X_i^u, Y_i^u$ for $u \in \{0,1\}^k$. For $1 \leq i \leq k$ we define

$$M_i = A_i^{0^l} \cup \bigcup_{\substack{v \in V(Q_l) \\ v \neq 0^l}} B_i^v \,.$$

For $1 \leq j \leq l$ we define

$$N_j = \bigcup_{\substack{u \in V(Q_k) \\ u \text{ odd}}} X_j^u \cup \bigcup_{\substack{u \in V(Q_k) \\ u \text{ even}}} Y_j^u \,.$$

Clearly, $M_1, \ldots, M_k, N_1, \ldots, N_l$ is a 1-factorization of $Q_{k+l}$. It remains to prove that $M_i \cup N_j$ forms a Hamiltonian cycle for every $i \in \{1, \ldots, k\}$, $j \in \{1, \ldots, l\}$. Let us show it with Figure 7.

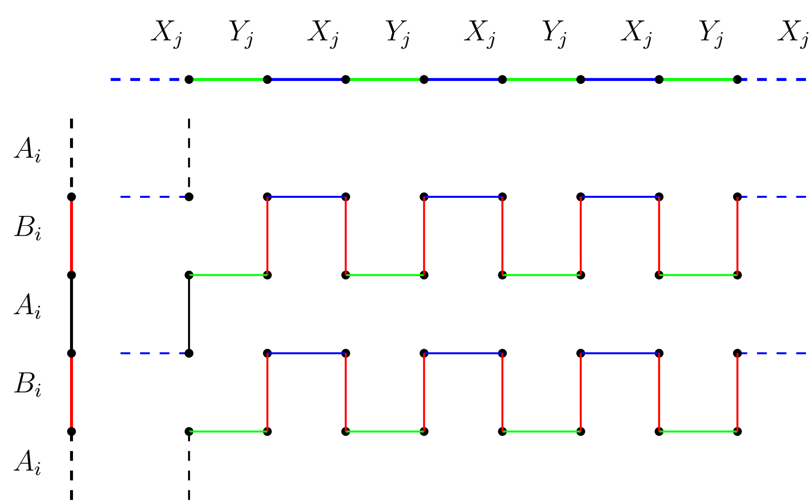


**Figure 7**: A Hamiltonian cycle $M_i \cup N_j$ in $Q_k \,\square\, Q_l$ from Hamiltonian cycles $A_i \cup B_i$ in $Q_k$ and $X_j \cup Y_j$ in $Q_l$.

■

It is known that a $k$-semi-perfect 1-factorization of $Q_{k+l}$ exists also for $k = 3$ and $l \neq 3$. A 3-semi-perfect 1-factorization of $Q_6$ is still unknown.

For a 1-factorization $\mathcal{F} = \{F_1, \ldots, F_n\}$ of a graph $G$ let $H(\mathcal{F})$ be the graph defined on 1-factors $F_1, \ldots, F_n$ as vertices where $F_i$ and $F_j$ are connected by an edge if and only if $F_i \cup F_j$ is a Hamiltonian cycle of $G$. Theorem 14 says that $Q_{k+l}$ for $k \neq 3$, $l \neq 3$ has a 1-factorization $\mathcal{F}$ with $H(\mathcal{F})$ being the complete bipartite graph $K_{k,l}$. The following theorem shows it is the best possible.

**Theorem 15 (Laufer [15])** *Let $G$ be a bipartite graph with both partities of size $n$ where $n$ is even and let $\mathcal{F}$ be its 1-factorization. Then $H(\mathcal{F})$ is bipartite.*

**Proof** Let $U, V$ be the partities of $G$. Each 1-factor $F_i$ of $G$ induces a bijection from $U$ to $V$. For $F_i, F_j \in \mathcal{F}$ let $\pi_{ji} = F_j^{-1} F_i$, so $\pi_{ji}$ is a permutation on $U$. Clearly, $\pi_{ii} = \mathrm{id}$, $\pi_{ij} = \pi_{ji}^{-1}$ and $\pi_{ij}\pi_{jk} = \pi_{ik}$. If $F_i F_j \in E(H(F))$, then $\pi_{ji}$ is a cycle of length $n$ ($F_i$ and $F_j$ form a Hamiltonian cycle), so it is an odd permutation ($\mathrm{sgn}(\pi_{ij} = -1)$ as $n$ is even.

Suppose $H(\mathcal{F})$ contains a cycle $F_{i_1} \ldots F_{i_k}$ of odd length. Then

$$1 = sgn(\pi_{i_1 i_1}) = sgn(\pi_{i_1 i_k} \pi_{i_k i_{k-1}} \ldots \pi_{i_2 i_1}) = \mathrm{sgn}(\pi_1 \pi_k) \prod_{j=2}^{k} \mathrm{sgn}(\pi_{i_j i_{j-1}}) = (-1)^k = -1\,,$$

which is a contradiction. ■

## Notes

Lemma 9 is a particular case of a more general result from Stong [21]. The used method is an adaptation of the method of Foregger [11], Aubert and Schneider [1].

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