Hypercube problems

Lecture 16

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1 Bootstrap percolation

In this lecture we focus on problems associated with bootstrap percolation. Motivation behind those problems is for example Conway's game of life (cellular automata), statistical mechanics, disease spreading etc.

Definition 1 *For a set $A \subseteq V(G)$ let $A_0 = A$ and for $t \geq 1$ let*

$$A_t = A_{t-1} \cup \{v \in V(G)\,;\, |N_G(v) \cap A_{t-1}| \geq r\}$$

*and let $\langle A \rangle = \bigcup_{t=0}^{\infty} A_t$ denote the closure of A. A **percolates** if $\langle A \rangle = V(G)$.*

We can simply imagine this using the motivation of studying spread of diseases. Let $A_G \subseteq V(G)$ be the set of "initially infected sites" in a graph G. In each time step, every "healthy" vertex that has at least r "infected" neighbors becomes infected as well. The variable parameter is called the *spreading threshold*.

Typically, the parameter r is 2 and G is some "lattice graph" (the $[n]^d$ grid), tree etc. If not specified otherwise, in this lecture we suppose that $r = 2$.

2 2-bootstrap percolation in Q_n

Definition 2 *A set $A \subseteq V(Q_n)$ is **closed** (or 2-closed) if every $x \in V(Q_n)$ has at most one neighbor in A, i.e. $A = \langle A \rangle$.*

Note that $\langle A \rangle$ is the intersection of all closed sets containing A. For example, all the subcubes are closed, two subcubes at distance at least 3 form a closed set.

Observation 3

- *If $A \subseteq Q_l \subseteq Q_n$ then $\langle A \rangle \subseteq Q_l$.*
- *For $x, y \in \{0, 1, *\}^n$ with $d(x, y) \leq 2$ we have $\langle Q^x \cup Q^y \rangle = Q^{x \vee y}$; where $d(x, y)$ is the number of coordinates i such that $x_i = 0$ and $y_i = 1$ (or vice versa), Q^x is the subcube prescribed by x, and*

$$(x \vee y)_i = \begin{cases} x_i & \text{if } x_i = y_i \\ * & \text{otherwise} \end{cases},$$

which means that $Q^{x \vee y}$ is the smallest subcube containing both Q^x and Q^y.

Proposition 4 *$A \subseteq V(Q_n)$ is closed if and only if A is a union of disjoint subcubes that are at distance at least 3 from each other.*

Proof Subcubes are closed sets, therefore for three subcubes that have distance at least 3 from each other, there is no vertex that is adjacent to two (or more) subcubes. This means that $A = \langle A \rangle$, informally speaking, the disease can spread nowhere.

Conversely, percolation process can be viewed as replacing two subcubes Q^x and Q^y with $d(x,y) \leq 2$ by the subcube $Q^{x \vee y}$ (see Observation 3). Since A is a union of subcubes of dimension 0, this process can be repeated until all the subcubes have distance at least 3 from each other. ■

Proposition 5 *The minimal size of a percolating set in Q_n under 2-bootstrap is*

$$m(Q_n, 2) = \left\lceil \frac{n}{2} \right\rceil + 1$$

Observation 6 *If Q^x has dimension k, Q^y dimension l, the distance $d(x,y) \leq 2$ then $Q^{x \vee y}$ has dimension less than or equal to $k + l + 2$.*

Proof We will prove this equality by proving the two inequalities separately.

$\leq$ Let us take 0^n and $\lceil \frac{n}{2} \rceil$ vertices in level 2 (the vertices in distance 2 from 0^n) so that they cover level 1. Now all the vertices in the first level surely have at least two "infected" neighbors and therefore, the first level is covered as well. In this manner all the levels will be eventually "infected" and therefore, the set percolates.

$\geq$ Let A be a percolating set of minimal size. The percolation process has at most $|A| - 1$ steps of replacing Q^x and Q^y by $Q^{x \vee y}$. Each replacement increases the dimension by at most 2 (see Observation 6) and at the end we obtain dimension n. Thus $n \leq 2|A| - 2$.

Therefore, the minimal size of a percolating set is $\lceil \frac{n}{2} \rceil + 1$. ■

Definition 7 *A subcube Q_l in Q_n is* ***internally spanned*** *by $A \subseteq V(Q_n)$ if $\langle A \cap Q_l \rangle = Q_l$.*

Theorem 8 (Balogh and Bollobás [1]) *Let A be a percolating set in Q_n. Then there is a nested sequence*

$$Q^{x_1}_{i_1=0} \subsetneq Q^{x_2}_{i_2} \subsetneq \ldots \subsetneq Q^{x_t}_{i_t} = Q_n$$

of internally spanned subcubes with $2_{i_j} + 2 \geq i_{j+1}$ for all j, $0 \leq j \leq t-1$.

Furthermore, for $j \geq 1$ each subcube $Q^{x_j}_{i_j}$ is spanned by two internally spanned subcubes: $Q^{x_{j-1}}_{i_{j-1}}$ and a subcube $Q_{m_{j-1}}$ with $m_{j-1} \leq i_{j-1}$ which is not a member of the sequence.

Remark The longest nested sequence is called a **building sequence**.

Proof Consider the process as before. It can be represented by a tree such as the one in Figure 1, which has subcubes of dimension 0 from A in the leaves. Internal vertex represent the hypercube $Q^{x \vee y}$ if Q^x and Q^y are represented by the children of the vertex.

From Observation 6 we already know that $2_{i_j} + 2 \geq i_{j+1}$ is satisfied for all j, $0 \leq j \leq t-1$. A branch in the tree (preferring the larger child) represents the nested sequence, from which the rest follows. ■

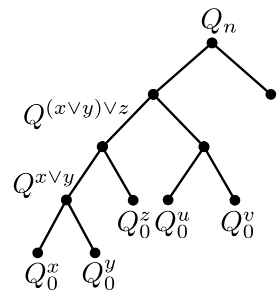

Figure 1: A tree representing the percolation steps

3 Largest minimal percolating sets

By minimal we mean minimal with respect to inclusion.

Definition 9 *Let $E(Q_n)$ denote the largest size of a percolating set $A \subseteq V(Q_n)$ such that $A \setminus \{v\}$ does not percolate for any $v \in A$. Such A is minimal with respect to inclusion.*

Lemma 10 (Riedl [8]) *For any n,*

- $E(Q_n) \geq n$,
- $E(Q_n) \geq E(Q_{n-1})$,
- $E(Q_n) \leq \max\{E(Q_{n-1}) + 1, 2E(Q_{n-4})\}$.

Proof The size of the percolating set has to be at least n because if we select all the n vertices at the first level, the set percolates. If we omit any of those vertices, the set does not percolate

Let A be a minimal (w. r. t. inclusion) percolating set in Q_{n-1} and $u \in A$. If u has no neighbor in A, we can safely add a neighbor of u from the new dimension to A, creating a larger minimal percolating set. If u has a neighbor in A, we can replace it by its neighbor, creating a minimal set of the same size in Q_n (see Figure 2).

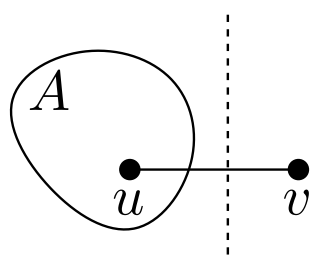

Figure 2: Enlarging the minimal percolating set A

Let $\langle P \cup R \rangle = Q_n$ be the last step in the building sequence for the minimal percolating set A. Let $\dim P \geq \dim R$ and suppose that we select P and R with largest $\dim P$ (over all building sequences).

Now we can divide the proof into 4 cases.

$\dim P = n-1$. By minimality of A there is exactly one vertex in R and $A \cap P$ is a minimal percolating set in P, therefore $|A| \leq E(Q_{n-1}) + 1$.

$\dim P = n-2$. The dimension of P implies that every vertex is in distance at most 2 from P. There must be a vertex v such that $d(v, P) = 2$ and $\langle P \cap v \rangle = Q_n$, otherwise we could extend P to a subcube of dimension $n-1$ (the previous case). Now we have $|A| \leq E(Q_{n-2}) + 1$.

$\dim P = n-3$. Because we selected the largest possible dimension of P, there cannot be a vertex of $A \cap R$ in distance less than two. This means that $A \cap R$ is contained in a subcube in distance 3 from P. Without loss of generality we can assume that P has the form $000 * \ldots *$ and therefore R has the form $111 * \ldots *$. Now we see that $d(P, R) = 3$, which contradicts the fact that A percolates (see Proposition 4). Thus this case does not occur.

$\dim P = n-4$. Similarly to the previous cases we see that $\dim R \leq n-4$. P and R are both minimally internally spanned, so $|A \cap P|, |A \cap P| \leq E(Q_{n-4})$. Therefore $|A| \leq 2E(Q_{n-4})$.

Hence we have proven that the lemma holds in every case. ■

Definition 11 *A minimal percolating set A in Q_n is **jagged** if $\langle A \setminus v \rangle$ is disjoint from the $(n-2)$-dimensional subcube $* \cdots * 00$ for all $v \in A$.*

Definition 12 *Let $E'(Q_n)$ be the largest size of a jagged set in Q_n.*

Lemma 13 (Riedl [8]) *For every $n > 5$ it holds that*

$$E'(Q_n) \geq 2E'(Q_{n-4}).$$

Proof Let A be a jagged minimal percolating set in Q_{n-4} of size $|A| = E'(Q_{n-4})$. We pick B_1 and B_2 as sets isomorphic to A in different subcubes. Without loss of generality suppose that B_1 is jagged in the subcube $* \ldots * 0001$ and B_2 is jagged in the subcube $* \ldots * 00 * * 10$.

Let $B = B_1 \cup B_2$. From Proposition 4 and the fact that $d(B_1, B_2) = 2$ we know that B percolates. Without loss of generality, suppose that we remove v from B_1. Because B_1 is jagged and minimal in Q_{n-4}, $\langle B_1 \setminus v \rangle$ is a union of subcubes at distance at least 3, and therefore they have at least one 1 in $(n-4)$-th or $(n-5)$-th coordinate. Thus, each subcube in $B_1 \setminus v$ will also have distance at least 3 from B_2, which means that $B \setminus v$ does not percolate. B is minimal, and it is also jagged because every vertex of $\langle B \setminus v \rangle$ has 01 or 10 in the last two coordinates.

We constructed a minimal jagged set B of size $2|A|$ in Q_n. ■

Lemma 14 (Riedl [8]) *There is a jagged set of size $n-1$ in Q_n for every $n \geq 4$, thus $E'(Q_n) \geq n-1$.*

Proof Let $A = \{e_n \oplus e_i \mid i = 1 \dots n-2\} \cup \{1 \dots 10\}$. Thanks to the added vertex, which is the only one without 01 at the end, A percolates. Therefore $\langle A \setminus \{1 \dots 110\} \rangle$ is disjoint from $* \dots *00$. If we omit any other vertex, by permuting the first $n-2$ coordinates, it is the same as if we omit $e_n \oplus e_1$. The set $\{e_n \oplus e_i \mid i = 2 \dots n-2\}$ combines to $0 * \dots * 01$, which is in distance 3 from $1 \dots 10$ and therefore $\langle A \setminus \{e_n \oplus e_i\} \rangle$ does not percolate. Moreover, it splits into two subcubes disjoint from $* \dots * 00$ which means that we have found a jagged set of size $n-1$. ■

Theorem 15 (Riedl [8]) *Let $1 \le r \le 4$ so that $n \equiv r(\mod 4)$. Then*

$$E(Q_n) = \begin{cases} n+1 & \text{if } 0 \le n \le 1, \\ n & \text{if } 2 \le n \le 10, \\ \left(1 + 2^{r-4}\right) 2^{\left\lfloor \frac{n+3}{4} \right\rfloor} & \text{if } n \ge 11. \end{cases}$$

Proof The proof follows from previous lemmas and analysis in small dimensions ■

Problem 1 Find $E(Q_n, r)$ for $r \ge 3$, the largest size of a minimal percolating set in Q_n with the spreading parameter r.

4 Maximal percolation time

For $A \subseteq V(Q_n)$ let $T(A) = \min\{t; A_t = \langle A \rangle\}$ (spreading time) and

$$M(n) = \max_{A : \langle A \rangle = Q_n} T(A)$$

Lemma 16 (Przykucki [7]) *For $x, y \in \{0, 1, *\}^n$, $k = \dim(x)$, $l = \dim(y)$, $m = \dim(x \vee y)$, $p = |\{i \mid x_i = y_i = *\}|$ the spreading time of $Q^x \cup Q^y$ is*

$$T(Q^x \cup Q^y) = \begin{cases} m - p & \text{if } d(x,y) = 2 \text{ and } (k,l) \neq (m-2, m-2), \\ m - p - 1 & \text{otherwise.} \end{cases}$$

Proof Omitted ■

Theorem 17 (Przykucki [7]) $M(1) = 0$, $M(2) = 1$, $M(3) = 3$, $M(4) = 5$ *and for $n \ge 5$*

$$M(n) = \max \begin{cases} M(n-2) + 4, \\ M(n-3) + 2n - 3. \end{cases}$$

Sketch of Proof The values for $n \le 4$ can be found by exhaustive search. For example for $n = 4$ we have $T(\{0000, 1100, 0111\}) = 5$. We will prove only that $M(n) \ge \max\{M(n-2) + 4, M(n-3) + 2n - 3\}$.

Consider two ways of infecting Q_n.

1. Let A be a set that internally spans the hypecube of dimension $n-2$ given by $* \dots * 00$ in time $T_{Q_{n-2}}(A) = M(n-2)$. Then

$$T_{Q_n}(A00 \cup \{0\dots011\}) = M(n-2) + 2$$

.

2. Let B be a set that internally spans the hypecube of dimension $n-3$ given by $* \dots * 000$ in time $T_{Q_{n-3}}(B) = M(n-3)$. Then

$$T_{Q_n}(B000 \cup \{0\dots0110, 1\dots1\}) = M(n-3) + 2n - 3$$

.

Detailed proof can be found in Przykucki [7]. ■

Corollary 18

$$M_n = \left\lfloor \frac{n^2}{3} \right\rfloor \text{ for all } n \geq 1.$$

5 Random bootstrap percolation

Each vertex $x \in (Q_n)$ is chosen to A independently with probability p. How to choose p (depending on n) if A should be likely to percolate?

Definition 19 *For* $0 \leq \alpha \leq 1$ *let*

$$p_\alpha(Q_n, r) = \inf\{p \mid \mathbb{P}(A \text{ percolates in } r\text{-neighbor bootstrap}) \geq \alpha\}$$

Definition 20 *The **critical probability** is defined as*

$$p_c(Q_n, r) = p_{1/2}(Q_n, r).$$

Why does it matter? Percolation is monotone [1] and symmetric [2] property, thus by theorem of Friedgut and Kalai [4] it has a sharp threshold, i.e. for every $0 < \varepsilon < \frac{1}{2}$

$$\lim_{n\to\infty} \frac{p_{1-\varepsilon}(Q_n, r) - p_\varepsilon(Q_n, r)}{p_c(Q_n, r)} = 0.$$

Theorem 21 (Balogh, Bollobás and Morris [3]) *Let λ be the smallest positive root of the equation*

$$\sum_{k=0}^{\infty} \frac{(-1)^k \lambda^k}{2^{k^2-k} k!} = 0,$$

[1] Adding a vertex to a percolating set gives a percolating set.

[2] For every two vertices x, y there is a permutation Π such that $\Pi(x) = \Pi(y)$ and $\Pi(A) = A$ where A is a percolating set.

so $\lambda \approx 1.166$. *Then*

$$\frac{16\lambda}{n^2}\left(1+\frac{\log n}{\sqrt{n}}\right)2^{-2\sqrt{n}} \le p_c(Q_n,2) \le \frac{16\lambda}{n^2}\left(1+\frac{5(\log n)^2}{\sqrt{n}}\right)2^{-2\sqrt{n}}$$

The lower bound is the probability with which the random set is likely not percolating, the upper bound is the probability with which the set percolates.

Problem 2 (Balogh, Bollobás and Morris [3]) Determine $1 \le \alpha \le 2$ (if it exists) so that for any $\varepsilon > 0$

$$\frac{16\lambda}{n^2}\left(1+\frac{(\log n)^{\alpha-\varepsilon}}{\sqrt{n}}\right)2^{-2\sqrt{n}} \le p_c(Q_n,2) \le \frac{16\lambda}{n^2}\left(1+\frac{(\log n)^{\alpha+\varepsilon}}{\sqrt{n}}\right)2^{-2\sqrt{n}}$$

6 Minimal size of an r-bootstrap percolating set

Definition 22 *Let* $m(Q_n)$ *denote the minimal size of a set that percolates under* r*-neighbor bootstrap process on* Q_n.

Proposition 23

$$m(Q_n,r) \le \frac{1+o(1)}{r}\binom{n}{r-1}$$

Sketch of Proof Let $n \ge 2r$, A_0 be the level $r-2$ and A be a system of r-sets so that every $(r-1)$-set is contained in at least one set of A, i. e. A covers the level $r-1$.

Rödl [9] showed that there is such A of size

$$\frac{1+o(1)}{r}\binom{n}{r-1}.$$

Then $A \cup A_0$ clearly percolates. ■

Remark Keevash [5] showed that (up to finitely many cases) an exact Steiner system exists under certain conditions on n and r. The percolating set of cardinality $\frac{1}{r}\binom{n}{r-1}+\binom{n}{r-2}$ yields

$$m(Q_n,r) \le \frac{n^{r-1}}{r!} + \frac{n^{r-2}(r+2)}{2r(r-2)!} + \mathcal{O}(n^{r-3}).$$

Theorem 24 (Morrison and Noel [6]) *For* $n \ge r \ge 1$ *it holds*

$$m(Q_n,r) \ge 2^{r-1} + \sum_{j=1}^{r-1}\binom{n-j-1}{r-j}\frac{j2^{j-1}}{r}.$$

Corollary 25

$$m(Q_n, r) \geq \frac{n^{r-1}}{r!} + \frac{n^{r-2}(6-r)}{2r(r-2)!} + \Omega(n^{r-3})$$

In particular, $m(Q_n, r) = \frac{1+o(1)}{r}\binom{n}{r-1}$

Theorem 26 (Morrison and Noel [6]) *For* $n \geq 3$*,*

$$m(Q_n, 3) = \left\lceil \frac{n(n+3)}{6} \right\rceil + 1$$

Problem 3 Does

$$\lim_{n\to\infty} \frac{m(Q_n) - \frac{n^{r-1}}{r!}}{n^{r-2}}$$

exist?

Problem 4 Is there a sufficiently large n such that

$$m(Q_n, r) = 2^{r-1} + \left\lceil \sum_{j=1}^{r-1} \binom{n-j-1}{r-j} \frac{j2^{j-1}}{r} \right\rceil ?$$

Problem 5 Determine $m(Q_n, 4)$ for all $n \geq 4$.

7 Majority Bootstrap Percolation

We will assume that n is even and $r = \lceil \frac{n}{2} \rceil$.

Theorem 27 (Balogh, Bollobás and Morris [2]) *Let* $\lambda \in \mathbb{R}$ *and* $p = p(n) = \frac{1}{2} - \frac{1}{2}\sqrt{\frac{\log n}{n}} + \frac{\lambda \log\log n}{\sqrt{n \log n}}$ *and let the elements of* $A \subseteq V(Q_n)$ *be chosen independently at random, each with probability* p*. Then in* ***majority bootstrap percolation***

$$\mathbb{P}(A \text{ percolates on } Q_n) \to \begin{cases} 0 & \text{if } \lambda \leq -2 \\ 1 & \text{if } \lambda > \frac{1}{2} \end{cases} \quad \text{as } n \to \infty.$$

In particular,

$$p_c\left(Q_n, \frac{n}{2}\right) \geq \frac{1}{2} - \frac{1}{2}\sqrt{\frac{\log n}{n}} + \frac{2\log\log n}{\sqrt{n\log n}}$$

for sufficiently large n *and*

$$p_c\left(Q_n, \frac{n}{2}\right) \leq \frac{1}{2} - \frac{1}{2}\sqrt{\frac{\log n}{n}} + \frac{\log\log n}{2\sqrt{n\log n}} + o\left(\frac{\log\log n}{2\sqrt{n\log n}}\right)$$

as $n \to \infty$*.*

References

[1] J. Balogh and B. Bollobás, *Bootstrap percolation on the hypercube*, Probability Theory and Related Fields, 134 (2006), 624–648.

[2] J. Balogh, B. Bollobás, and R. Morris, *Majority bootstrap percolation on the hypercube*, Combinatorics, Probability & Computing, 18 (2009), 17–51.

[3] J. Balogh, B. Bollobás, and R. Morris, *Bootstrap percolation in high dimensions*, Combinatorics, Probability and Computing, 19 (2010), 643–692.

[4] E. Friedgut and G. Kalai, *On a packing and covering problem*, Every monotone graph property has a sharp threshold, 124 (1996), 2993–3002.

[5] P. Keevash, *The existence of designs*, (2014). arXiv: 1401.3665v1.

[6] N. Morrison and J. A. Noel, *Extremal bounds for bootstrap percolation in the hypercube*, (2015). arXiv: 1506.04686v1.

[7] M. Przykucki, *Maximal percolation time in hypercubes under 2-bootstrap percolation*, Electronic Journal of Combinatorics, 12 (2012).

[8] E. Riedl, *Largest minimal percolating sets in hypercubes under 2-bootstrap percolation*, Electronic Journal of Combinatorics, 17 (2010).

[9] V. Rödl, *On a packing and covering problem*, European Journal of Combinatorics, 6 (1985), 69–78.