Hypercube problems

Lecture 17

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1 Symmetric chain decomposition

As the title suggests, in this lecture we will explore symmetric chain decompositions (SCD). Let us start by defining what an SCD is and by proving that there indeed exists an SCD.

Definition 1 *A* symmetric chain *in a Boolean lattice $\mathcal{B}_n = (\mathcal{P}([n]), \subseteq)$ is a sequence $X_k \subset X_{k+1} \subset \cdots \subset X_{n-k}$ such that $|X_i| = i$ for every $k \leq i \leq n-k$. A* symmetric chain decomposition *(SCD) is a collection $\mathcal{C}$ of disjoint symmetric chains that covers $\mathcal{B}_n$.*

Basically, a symmetric chain is nothing more than a path in a hypercube Q_n from level k to level $n-k$, that only "goes up".

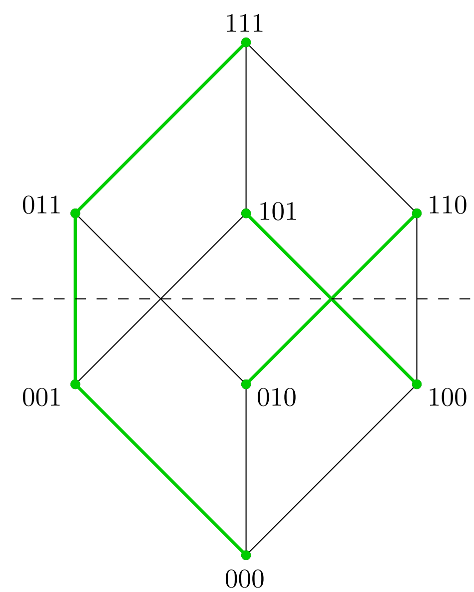


Figure 1: An example of a symmetric chain decomposition of Q_3. Symmetric chains are marked by green color.

Proposition 2 *For every $n \geq 1$, $\mathcal{B}_n$ has an SCD.*

We provide three ways to describe construction of the same SCD of $\mathcal{B}_n$.

Proof (Recursive construction) Clearly, $\mathcal{B}_1$ has an SCD. Now let us assume we have a decomposition $\mathcal{C}$ of $\mathcal{B}_n$. Now, for a symmetric chain $C \in \mathcal{C}$ we take two symmetric chains C', C'' in $\mathcal{B}_{n+1}$ as follows:

$$C' = (X_k, X_{k+1}, \ldots, X_{n-k}, X_{n-k} \cup \{n+1\}),$$
$$C'' = (X_k \cup \{n+1\}, X_{k+1} \cup \{n+1\}, \ldots, X_{n-k-1} \cup \{n+1\}),$$

where $C = (X_k, \ldots, X_{n-k})$. Both C' and C'' are indeed symmetric chains $\mathcal{B}_{n+1}$.

This gives us an SCD for $\mathcal{B}_{n+1}$ in the form of $\{C', C'' \,|\, C \in \mathcal{C}\}$ which satisfies the definition of SCD because $\mathcal{C}$ is an SCD for $\mathcal{B}_n$. Note that $\mathcal{C}''$ can be an empty sequence in which case we do not include it into the decomposition. As you can see in Figure 2, we just split Q_{n+1} along the $(n+1)$-st direction, recursively built SCD for both subcubes, then we extend chains in the subcube Q_n^0 from top vertices to their neighbors in Q_n^1 and shorten chains Q_n^1 by removing their top vertices. ■

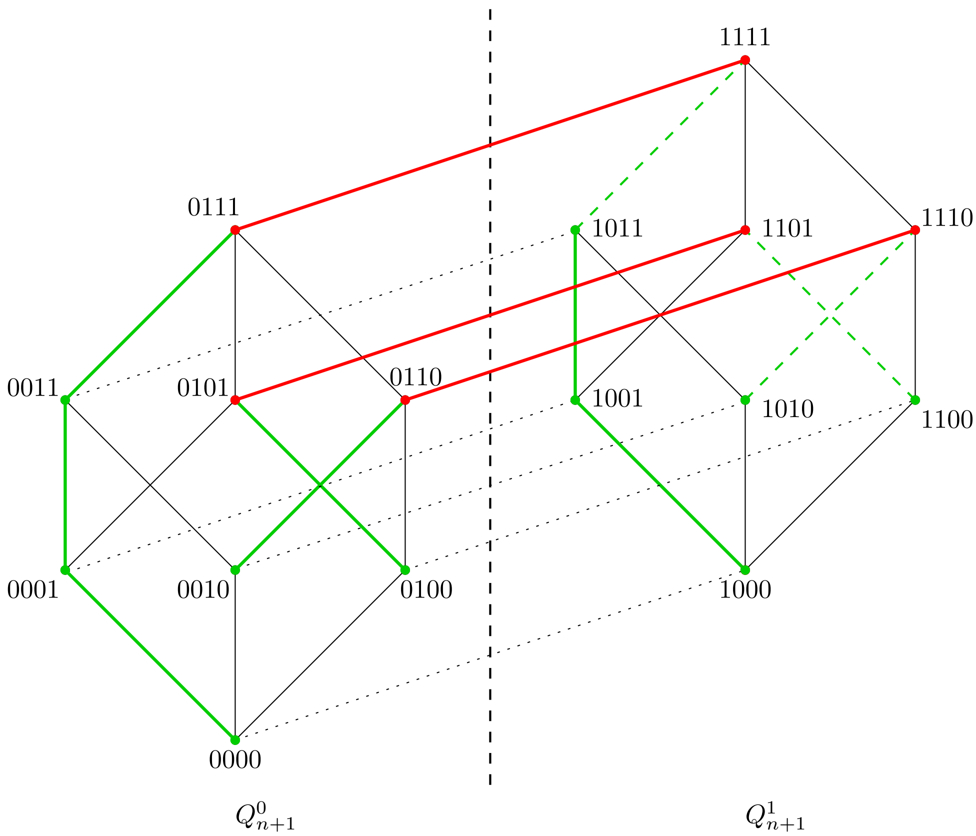


Figure 2: Recursive construction of an SCD for $n = 3$. We split Q_4 along the direction 4 and build SCD for both subcubes (green color). Then we add endpoints of chains in Q^0_{n+1} to respective chains in Q^1_{n+1} (red edges) and delete those vertices from chains in Q^1_{n+1} (dashed green edges).

For those readers who are not yet convinced or who aren't quite fond of such recursive constructs, we present another, perhaps more algorithmic, construction of SCD.

Proof (Greedy lex. matching) First, let us define lexicographical ordering of $\mathcal{P}([n])$ by $A <_{\text{lex}} B$ iff $\min(A \Delta B) \in B$. This exactly corresponds to lexicographical ordering of characteristic vectors of sets in $\mathcal{P}([n])$.

We construct an SCD as follows. For every level k, $0 \leq k \leq n-1$, we go through vertices on level k in lexicographical order (starting with the smallest one) and match them with their lexicographically smallest unmatched neighbor on level $k+1$. See the example in Figure 3.

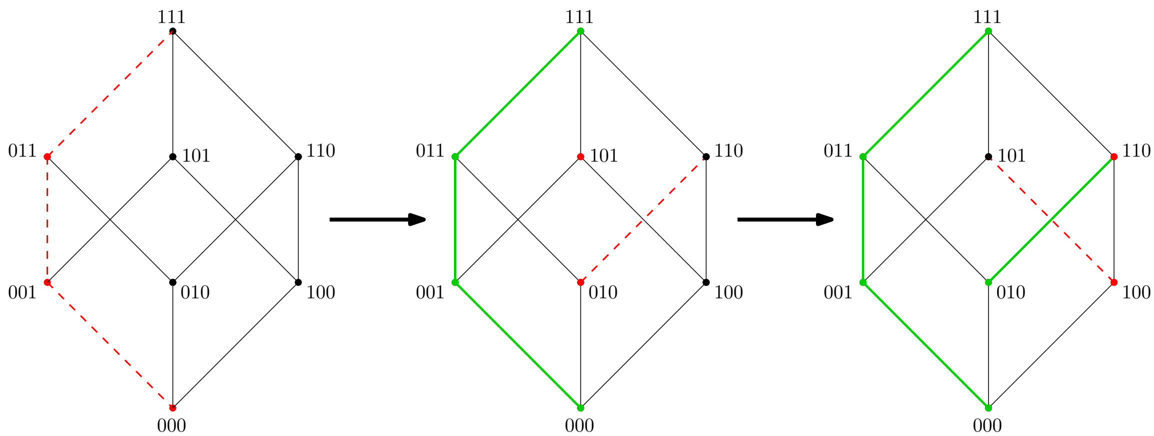


Figure 3: An example of greedy lexicographical matching algorithm on Q_3. In each phase we take lexicographically smallest, not yet processed, vertex in every layer (marked red) and try to match it with its lexicographically smallest unmatched neighbor in the layer above (red dashed lines). These matched vertices will form the chains in SCD (green).

Obviously, this algorithm creates a collection of disjoint chains (since every vertex is matched to at most one vertex from the previous level and at most one vertex from the next level) that covers whole $\mathcal{P}([n])$ (an isolated vertex is also a chain). It remains to show that these chains are symmetric. Instead of proving this directly, we just show that this construction gives the same chains as the construction from the previous proof.

The argument goes by induction. For $n = 1$ both constructions obviously give the same decomposition. Now let the SCDs be the same for $n \geq 1$ and let us look at $\mathcal{B}_{n+1}$. We split respective hypercube Q_{n+1} along the direction $n+1$ into Q^0_{n+1} and Q^1_{n+1}.

Note that all the vertices of Q^0_{n+1} are lexicographically smaller than all the vertices in Q^1_{n+1}. Thus, by induction, the matching algorithm produces the same chains in Q^0_{n+1} as the recursive construction. Then, the unmatched vertices are the top vertices in these chains, which start with 01. Their lexicographically smallest neighbors in the next level are from Q^1_{n+1}, hence they can be matched with them. This corresponds to adding the red edges (in Figure 2) as in the recursive construction. Furthermore, the matching algorithm produces the same chains in the Q^1_{n+1} as the recursive construction in Q_n, except that the vertices covered by red edges are already taken. This corresponds to removing the green edges, which effectively leads to the previous recursive construction. ■

And finally, to convince even the most skeptical readers, we describe the construction once more using a (perhaps surprising) trick.

```
00110100110
     ↓
001()()01()
     ↓
00(()())1()
```

```
11(()())1()
     ↕
01(()())1()
     ↕
00(()())1()
     ↕
00(()())0()
```

Figure 4: An example of pairing 1's and 0's for $x = 00110100110$. On the left it is shown how pairing is obtained (paired digits are substituted by appropriate parenthesis). On the right the symmetric chain containing x is shown.

Proof (Pairings of 1's and 0's) Let x be the characteristic vector of $X \in \mathcal{B}_n$. We pair 1's in x with closest 0's as if they we were parenthesis – one corresponds to opening parenthesis "(" and zero to ")". More precisely, we can imagine that we are doing pairing in phases – in i-th phase we try to pair all unpaired 1's with unpaired 0's that are exactly i positions from them to the right. An example can be seen in the Figure 4.

To get symmetric chain containing X, we simply define how to "go up" and "go down" along the chain. To go up we take the rightmost unpaired zero in x and flip it to one, to go down we flip the leftmost unpaired one. This obviously defines symmetric chain. It remains to observe that every two chains we create this way are either identical or disjoint. But that is easy – flipping the rightmost zero or leftmost one can never create a new one-zero pair. ■

2 The Littlewood-Offord problem

A trivial application of the symmetric chain decomposition is a proof of Sperner's theorem. Let us recall that a family $\mathcal{F} \subseteq \mathcal{B}_n$ is independent if for every $A, B \in \mathcal{F} : A \not\subset B$.

Theorem 3 (Sperner) *The maximum size of an independent family over n elements is $\binom{n}{\lfloor n/2 \rfloor}$.*

Proof All the vertices on the level k in $\mathcal{B}_n$ give an independent family of size $\binom{n}{k}$, so the level $\lfloor n/2 \rfloor$ gives the independent family of the desired size. To show that there is not a larger independent family, we consider an SCD $\mathcal{C}$ of $\mathcal{B}_n$. Since $\mathcal{C}$ covers $\mathcal{B}_n$, there can not be more than $|\mathcal{C}|$ independent sets in $\mathcal{B}_n$ (otherwise two of them would be part of the same chain). But $|\mathcal{C}| = \binom{n}{\lfloor n/2 \rfloor}$ since every chain in $\mathcal{C}$ intersects any level of $\mathcal{B}_n$ at most once. ■

Now, we show a more intricate application – we will use SCD to solve the *Littlewood-Offord problem.*

Problem 4 (Littlewood-Offord) *Let $x_1, x_2, \ldots, x_n \in B$, where B is a normed space, such that $\|x_i\| \geq 1$ for every $i \in [n]$. For $A \subseteq [n]$, $A \neq \emptyset$, let us denote $x_A = \sum_{i \in A} x_i$ and let $x_\emptyset = 0$. What is the maximum size of a family $\mathcal{F} \subseteq \mathcal{B}_n$ such that for every $A, B \in \mathcal{F} : \|x_A - x_B\| < 1$?*

Claim 5 *Let x_A for $A \subseteq [n]$ be defined as above and let $\mathcal{F} \subseteq \mathcal{B}_n$. If for every $A, B \in \mathcal{F}$: $\|x_A - x_B\| < 1$, then it holds that $|\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor}$.*

The proof has a similar structure as the proof of Sperner's theorem. We will partition $\mathcal{B}_n$ into families such that every family is "sparse", i.e. vectors x_A for A in the family are far away from each other. This means that our family $\mathcal{F}$ can be only as large as the number of families in the partition by argument similar to the one used in the proof of Sperner's theorem. By choosing the partition that can be mapped to an SCD, we will get the desired estimate.

Now we will define what do we mean by a sparse partition and its correspondence to an SCD.

Definition 6 *We say that $\mathcal{D} \subseteq \mathcal{B}_n$ is* sparse *is for every distinct $A, B \in \mathcal{D}$: $\|x_A - x_B\| \geq 1$. Moreover, a partition $\mathbb{D}$ of $\mathcal{B}_n$ is* symmetric *if there exists a bijection $h : \mathbb{D} \to \mathcal{C}$, where $\mathcal{C}$ is an SCD of $\mathcal{B}_n$, such that $|h(\mathcal{D})| = |\mathcal{D}|$ for every $\mathcal{D} \in \mathbb{D}$.*

We will also need a notion of supporting functional.

Definition 7 *A supporting functional for $y \in B$ is a continuous linear map $f : B \to \mathbb{R}$ such that $\|f\| = 1$ and $f(y) = \|y\|$. A norm of f is defined as $\|f\| = \sup_{x \in B} f(x)/\|x\|$.*

Note that $\|f\| = 1$ implies $f(x) \leq \|x\|$ for every $x \in B$. As an example, consider Euclidean space $B = \mathbb{R}^m$ and $y = (a, 0, \ldots, 0)$ for $a \geq 1$. Then the mapping $f(z_1, z_2, \ldots, z_m) = z_1$ is a supporting functional for y.

Proof of Claim 5: We will prove by induction that $\mathcal{B}_n$ has a symmetric partition into sparse sets. The case for $n = 1$ is trivial. Now assume that we have such partition $\mathbb{D}$ for $n-1$, $n > 1$. Let f be a supporting functional for x_n – its existence is granted by Hahn-Banach theorem. Let us choose arbitrary an $\mathcal{D} \in \mathbb{D}$ and denote $\mathcal{D} = \{A_1, A_2, \ldots, A_k\}$, also let $l \in [k]$ be such that $f(x_{A_l}) \geq f(x_{A_i})$ for every $i \in [k]$. We define

$$\mathcal{D}' = \{A_1, A_2 \ldots, A_k, A_l \cup \{n\}\},$$

$$\mathcal{D}'' = \{A_1 \cup \{n\}, A_2 \cup \{n\} \ldots, A_{l-1} \cup \{n\}, A_{l+1} \cup \{n\}\, A_k \cup \{n\}\}$$

and we claim that $\mathbb{D}' = \{\mathcal{D}', \mathcal{D}'' \,|\, \mathcal{D} \in \mathbb{D}\}$ is the desired symmetric partition into sparse sets.

Obviously, $\mathcal{D}''$ is sparse, since $\mathcal{D}$ is sparse. To prove sparsity of $\mathcal{D}'$ we need to show that $\|x_{A_l \cup \{n\}} - x_{A_i}\| \geq 1$ for every $i \in [k]$. From properties of f and l we have:

$$\begin{aligned} \|x_{A_l \cup \{n\}} - x_{A_i}\| &\geq f(x_{A_l \cup \{n\}} - x_{A_i}) \\ &\geq f(x_{A_l}) + f(x_n) - f(x_{A_i}) \\ &\geq f(x_n) = \|x_n\| \geq 1, \end{aligned}$$

where the first inequality follows from $\|f\| = 1$, the second follows from linearity of f and the third one from the choice of l.

To show the symmetry of $\mathbb{D}'$, we just notice that our construction of $\mathbb{D}'$ is almost identical to the recursive construction of SCD from the previous section. That is, if $\mathcal{D} \in \mathbb{D}$ maps to some symmetric chain $C \in \mathcal{C}$ (where $\mathcal{C}$ is the SCD from symmetry of $\mathbb{D}$), we can map $\mathcal{D}'$ to C' and $\mathcal{D}''$ to C'' using the notation from the recursive construction.

This gives the upper bound $|\mathcal{F}| \leq |\mathcal{C}| = \binom{n}{\lfloor n/2 \rfloor}$, since $|\mathbb{D}| = |\mathcal{C}|$ and $\mathcal{F}$ may intersect every set in $\mathbb{D}$ in only one point. ■

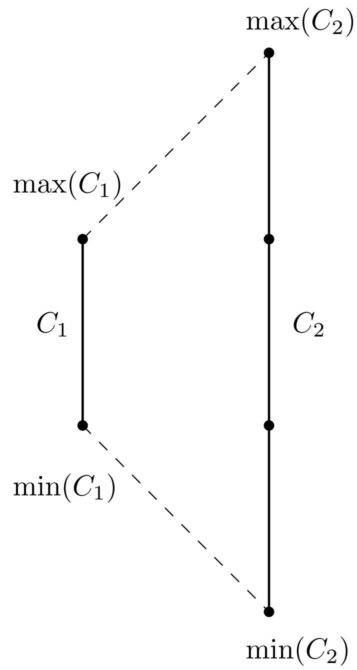


Figure 5: An example of a covered pair C_1, C_2.

3 Extending SCD into Hamiltonian cycle

From a graph theoretical point of view, a symmetric chain decomposition is a decomposition of Q_n into a set of disjoint paths (with some special properties). The natural question is whether it is possible to extend such decomposition into a Hamiltonian cycle by adding some edges. In this section we will show that for SCD constructed similarly as in the first section, it is indeed possible.

The construction of a Hamiltonian cycle will proceed inductively as is usual for hypercube problems, but we will need a Hamiltonian cycle with some special properties.

Definition 8 *Let $\mathcal{C}$ be an SCD of $\mathcal{B}_n$ and let us denote* $\max(C)$*, resp.* $\min(C)$ *the maximal, resp. minimal element of a chain $C \in \mathcal{C}$ with respect to inclusion. We say that $A \in \mathcal{B}_n$* covers *$B \in \mathcal{B}_n$ if $B \subset A$ and $|A \setminus B| = 1$, i.e. there is an "up" edge from B to A in $\mathcal{B}_n$. We call two chains $C_1, C_2 \in \mathcal{C}$ a* covered pair *if* $\max(C_2)$ *covers* $\max(C_1)$ *and* $\min(C_1)$ *covers* $\min(C_2)$.

Definition 9 *We say that an SCD $\mathcal{C}$ of $\mathcal{B}_n$ strongly extends to a Hamiltonian cycle if the order, in which Hamiltonian cycle traverses the chains of $\mathcal{C}$, gives partitioning of $\mathcal{C}$ into covered pairs and chains of size two.*

Theorem 10 *For every $n \geq 2$, $\mathcal{B}_n$ has an SCD that strongly extends into a Hamiltonian cycle.*

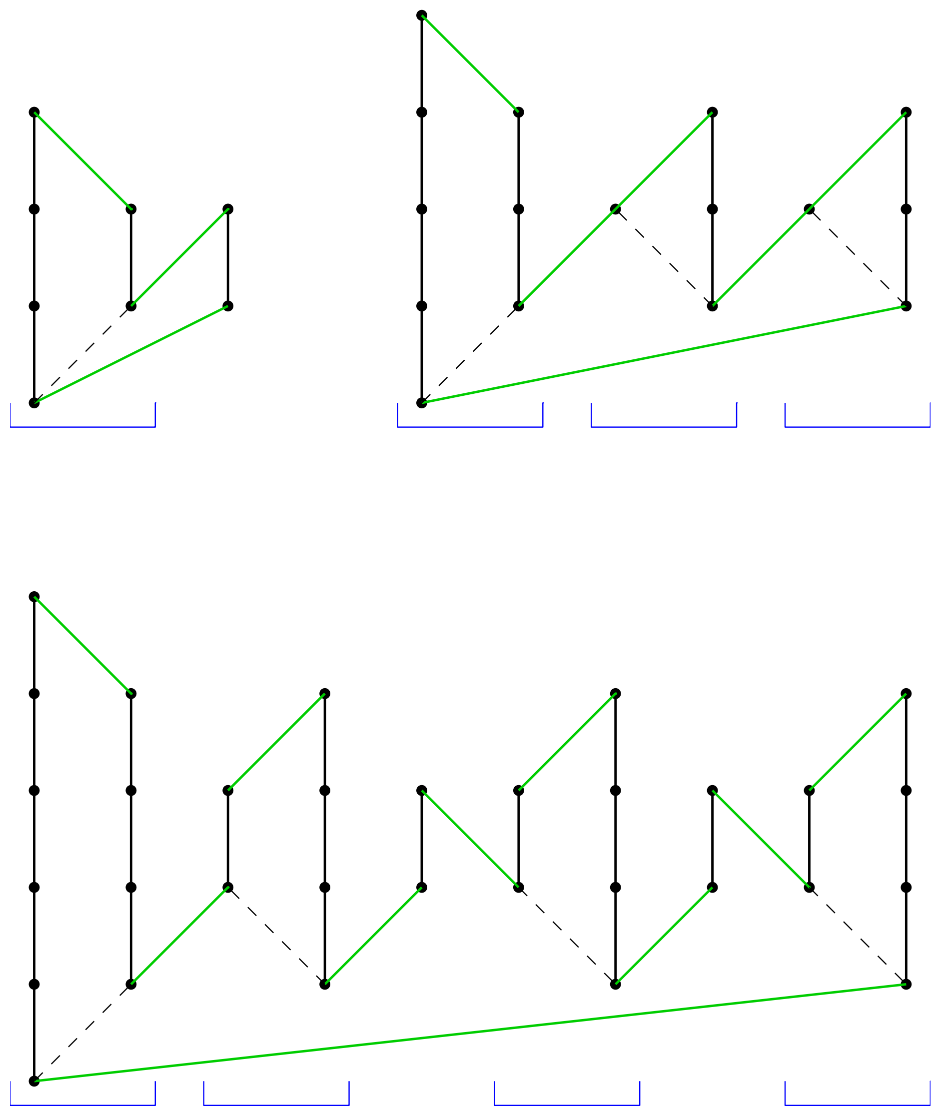


Figure 6: Examples of SCD that strongly extend into a Hamiltonian cycle for $\mathcal{B}_3, \mathcal{B}_4, \mathcal{B}_5$. Covered pairs are marked by a blue underline.

Proof As usual, we will use induction. For $n = 2$ such SCD obviously exists. Now suppose we want to construct SCD for $\mathcal{B}_n$, $n > 2$. We split $\mathcal{B}_n$ along the n-th direction and consider the same SCD $\mathcal{C}$ for both subcubes of dimension $n-1$. Now we take the partition $\mathcal{C}$ into covered pairs and single chains and distinguish three possible situations:

a) A covered pair C_1, C_2, such that $|C_2| = |C_1| + 2 \geq 4$.

b) A covered pair C_1, C_2, such that $|C_2| = |C_1| + 2 = 3$.

c) A single chain C (with $|C| = 2$)

For all these arrangements of chains we show how to modify their copies from 0- and 1-copy of $\mathcal{B}_{n-1}$ to get the chains in $\mathcal{B}_n$. In the cases a) and b) we also need to distinguish whether the Hamiltonian cycle "enters" C_2 in its maximal or minimal element. In this case, a picture is worth a thousand words, so we show the construction in Figure 7.

In all three situations, the construction preserves the endpoints of subpaths of the Hamiltonian cycle in the 0-copy of $\mathcal{B}_{n-1}$. Thus, all subpaths in $\mathcal{B}_n$ we constructed can be connected in the same way as the subpaths in the 0-copy of $\mathcal{B}_{n-1}$. The symmetry of the constructed chains follows from the symmetry of the original chains in $\mathcal{B}_{n-1}$. ■

Let us note that this construction can be extended to a Cartesian product of ranked posets [3]. Also it can be shown that the Hamiltonian cycle obtained by this construction has a minimal number of "peaks" – the vertices such that both their predecessor and successor on the cycle are on the same level (above or below the vertex). This is exactly the opposite of the monotone Gray code, which has a maximal number of peaks.

4 Venn Diagrams from SCDs

In the final section of these notes, we show how to use SCDs to construct Venn diagrams with some special properties. We assume that the reader is indeed familiar with Venn diagrams but let us state its definition more formally to make sure it is crystal clear what objects we are talking about.

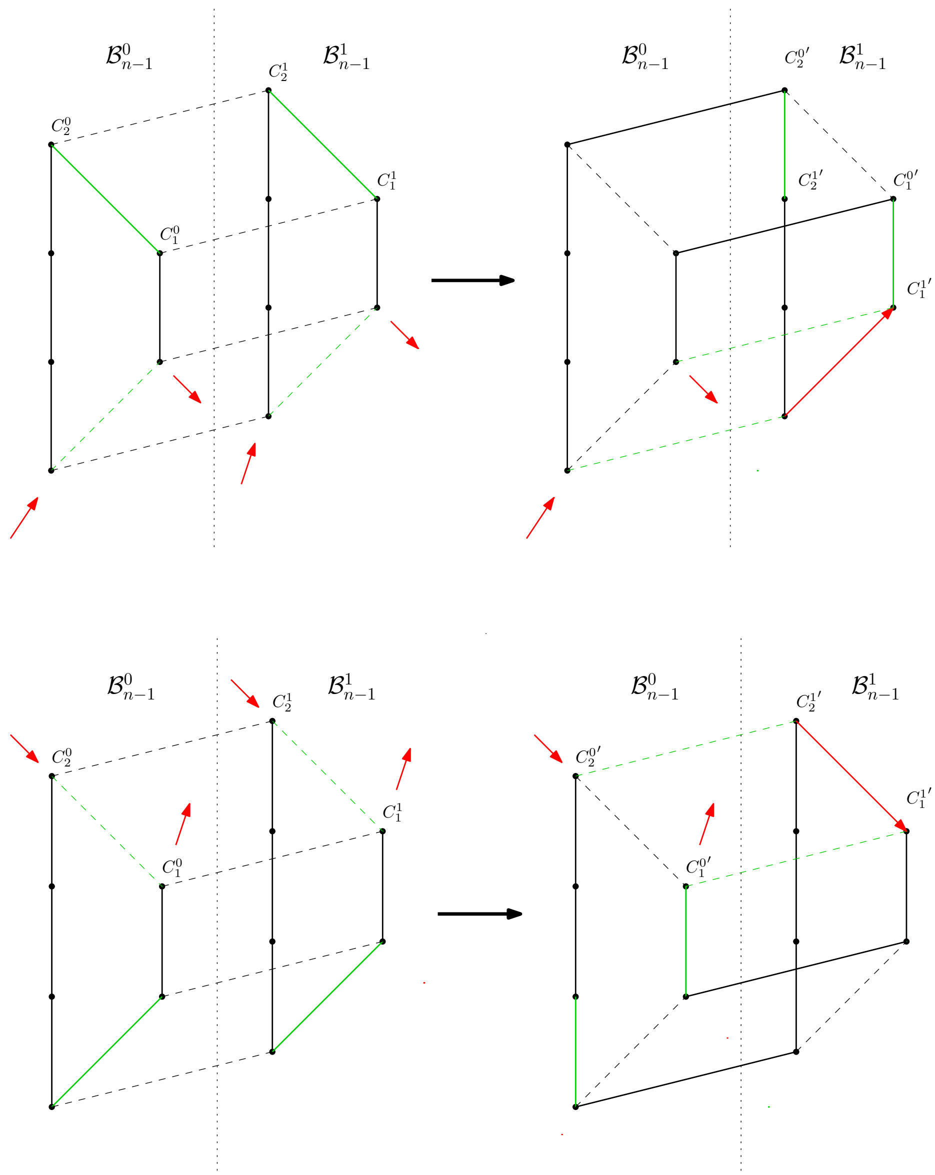
Definition 11 *An n-Venn diagram is a collection of n simple closed curves $\gamma_1, \dots, \gamma_n$ in the plane – interior* $\mathrm{int}(\gamma_i)$ *of γ_i represents the i-th set, while the exterior* $\mathrm{ext}(\gamma_i)$ *represents everything outside i-th set – such that for every $S \subseteq [n]$*

$$I_S = \bigcap_{i \in S} \mathrm{int}(\gamma_i) \cap \bigcap_{i \notin S} \mathrm{ext}(\gamma_i)$$

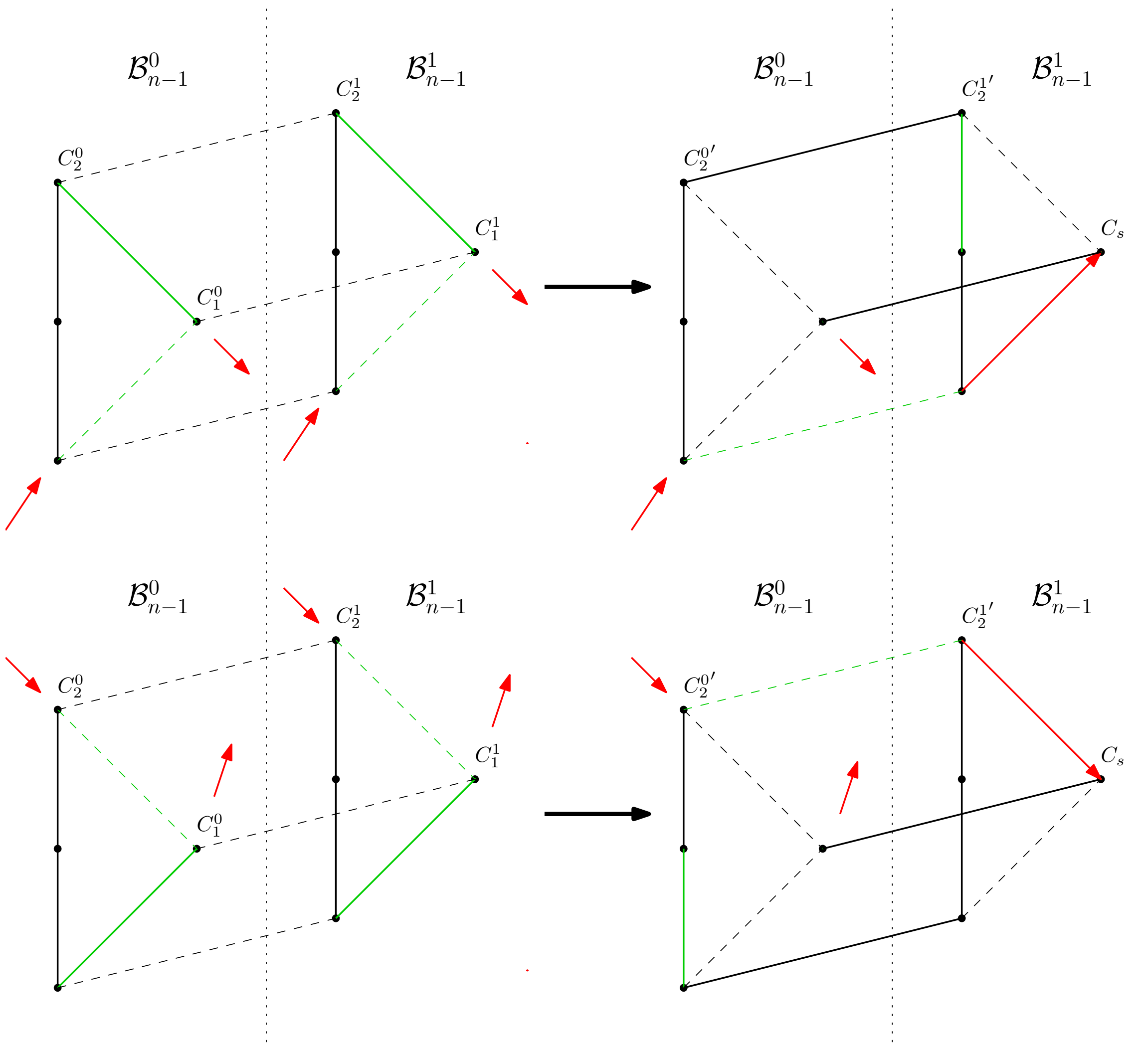
is nonempty and connected (the region I_S represents the intersection of sets in S).

We will be interested in monotone Venn diagrams that have as least vertices as possible. The vertex of a Venn diagram is a point in the plane where two or more γ_i's intersect. Thus we can see a Venn diagram as a plane graph with the vertices of its geometric dual corresponding to subsets of $[n]$.

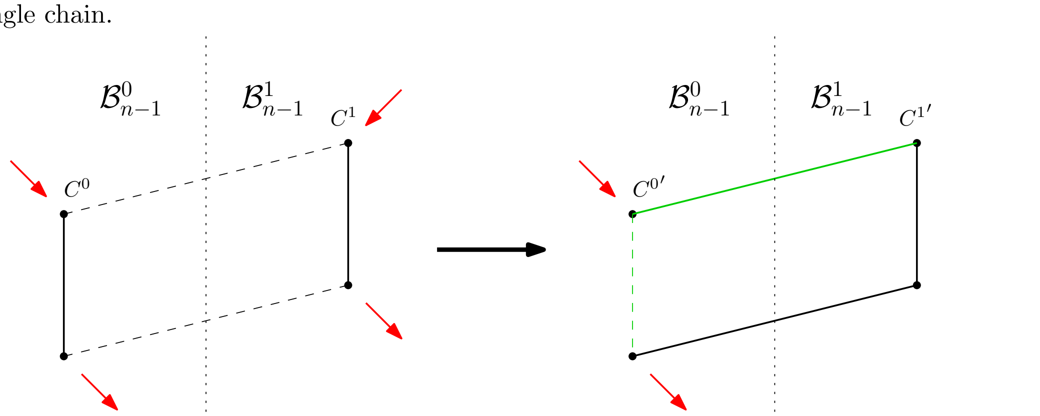
Figure 7: A construction of SCD that strongly extends to a Hamiltonian cycle for $\mathcal{B}_n$ from two SCDs for $\mathcal{B}_{n-1}$. Chains are marked as solid black lines and points. Edges between top and bottom vertices of covered pairs are marked by green color – the thick solid ones are those used by a Hamiltonian cycle. Red arrows show where Hamiltonian cycle enters or leaves the covered pairs or single chains. Notice that in all three cases we preserve endpoints of subpath of the Hamiltonian cycle in the 0-copy of $\mathcal{B}_{n-1}$.



Case a) A construction for a covered pair C_1, C_2 with $|C_2| = |C_1| + 2 \geq 4$. Two chains are extended by "stealing" a vertex from their copy in the other subcube, thus leaving us with two covered pairs.



Case b) A construction for a covered pair C_1, C_2 with $|C_2| = |C_1| + 2 = 3$. Chains of length 3 are changed into a covered pair in a similar way as in a), but the chains of length 1 are joined together into a single chain.



Case c) A construction for a single chain C. We end up with a covered pair.

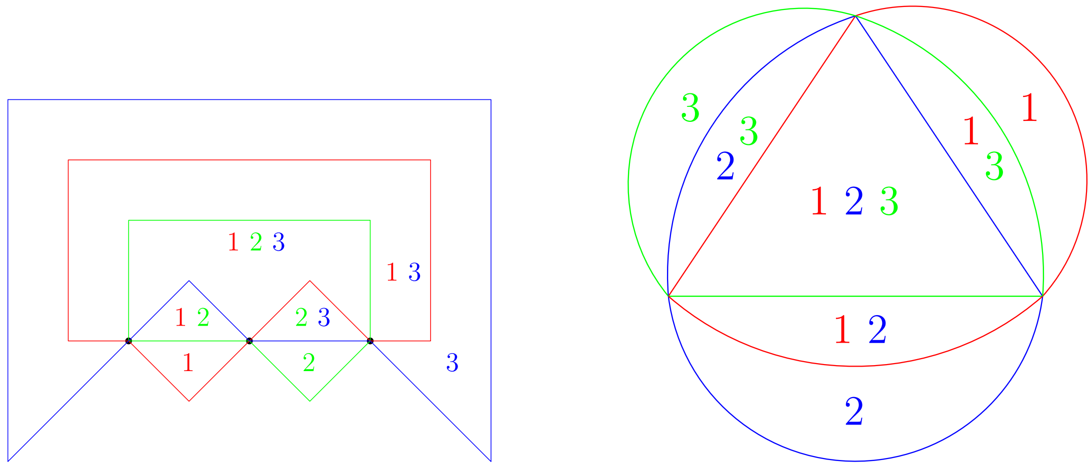


Figure 8: On the left there is a monotone Venn diagram for $n = 3$. On the right there is a convex Venn diagram that is isomorphic to the diagram on the left. Note, that it is also monotone.

Definition 12 *We say that a Venn diagram is* monotone *if for every $S \subseteq [n]$, $|S| = l$, the region corresponding to S is adjacent*[1] *to some region that corresponds to a subset of size $l - 1$ (if $l > 0$) and to some region that corresponds to a subset of size $l + 1$ (if $l < n$).*

For a motivation of what follows we state the following facts [2].

Fact 13 *A Venn diagram is isomorphic to a convex Venn diagram (that is, every* $\mathrm{int}(\gamma_i)$ *is convex) if and only if it is monotone.*

Fact 14 *Every monotone Venn diagram has at least $\binom{n}{\lfloor n/2 \rfloor}$ vertices.*

Now we show that the lower bound stated in Fact 14 is tight.

Theorem 15 *There is a monotone Venn diagram on $\binom{n}{\lfloor n/2 \rfloor}$ vertices for every $n \geq 1$.*

Proof Let $\mathcal{C}$ be an SCD from the first section. Then for every chain $C \in \mathcal{C}$ with $|C| < n+1$, there exists a chain $\pi(C)$ such that $\min(C)$ covers some element of $\pi(C)$ and $\max(C)$ is covered by some element of $\pi(C)$ (in fact, C and $\pi(C)$ form a covered pair, although this may not be true when we use similar construction for a different poset). Now let T be a *chain cover tree* – a rooted tree whose vertices are chains in $\mathcal{C}$ and C is a successor of D if $D = \pi(C)$ (see Figure 9). We consider a plane graph P, vertices of P are elements of $\mathcal{B}_n$ and there are two types of edges – those given by chains in SCD and those between chains given by π. Chains in P (i.e. the vertices and edges corresponding to the chains in C) are drawn in preorder of the chain cover tree. You can see an example in Figure 9.

[1] Two regions are adjacent if intersection of their boundaries has a nonzero length.

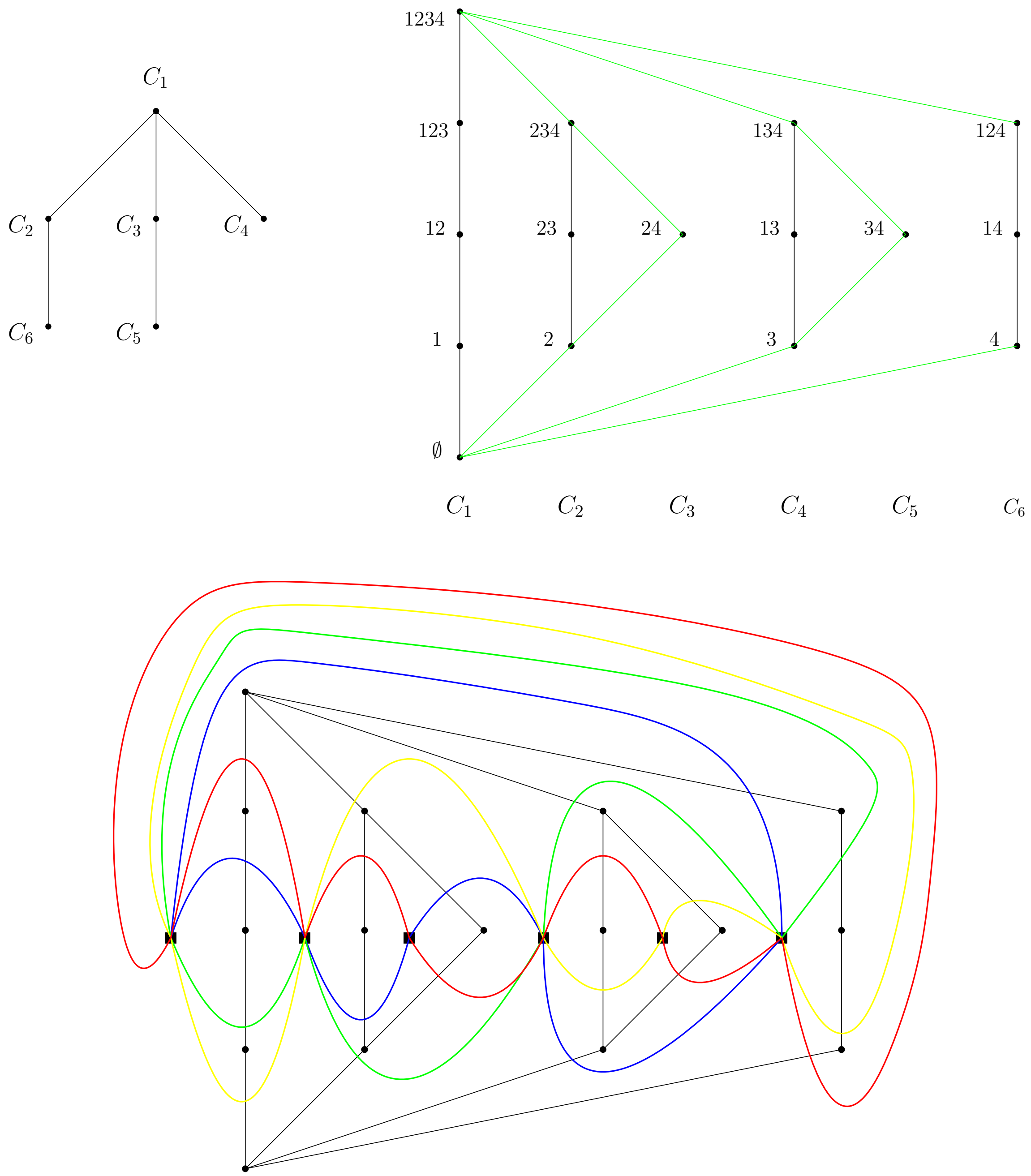


Figure 9: Construction of the monotone Venn diagram for $n = 4$. The top part shows the cover tree of the standard SCD of $\mathcal{B}_4$ (on the left) and the graph P derived from it (on the right). The lower part is the Venn diagram itself, vertices of P^* (and of the Venn diagram) are shown as black squares, edges of P^* are colored such that one color forms one S_i in the diagram. Notice that the vertices of P exactly correspond to the regions of the diagram.

Let P^* be a geometric dual of P. We claim that it corresponds to the desired monotone Venn's diagram. Clearly, P^* has $\binom{n}{\lfloor n/2 \rfloor}$ vertices, since each pair of chains $C, \pi(C)$ forms a face (plus one outerface for the longest chain, that has no $\pi(C)$). For $i \in [n]$, let S_i be the set of all edges in the direction i in P (note that P is a subgraph of a hypercube). Obviously, S_i is a *bond* in P (a minimal set of edges that disconnects P). Thus, the corresponding dual edges S_i^* in P^* form a cycle. Moreover, a region $\cap_{i \in A} \operatorname{int}(S_i^*)$ corresponds to a vertex A in P, showing that P^* is indeed a Venn diagram. Finally, since every vertex in P (except for $\emptyset$ and $[n]$) has both up and down edges, P^* is also monotone. ■

Let us note that one can similarly construct a symmetric Venn diagram from an SCD of necklace-representative subposet of B_n if n is prime [2].

References

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[2] Jerrold Griggs, Charles E Killian, and Carla D Savage. Venn diagrams and symmetric chain decompositions in the boolean lattice. *the electronic journal of combinatorics*, 11(1):2, 2004.

[3] Noah Streib and William T Trotter. Hamiltonian cycles and symmetric chains in boolean lattices. *Graphs and Combinatorics*, 30(6):1565–1586, 2014.