	Hypercube problems	
	Lecture 20	
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1 Sensitivity and block sensitivity

A Boolean function $f : \{0,1\}^n \to \{0,1\}$ corresponds to an induced subgraph H of Q_n, where $V(H) = \{x \mid f(x) = 1\}$.

Definition 1 *The* (local) sensitivity *of a function f on x is defined as $s(f,x) = |\{i \in [n] \mid f(x) \neq f(x \oplus e_i)\}|$. The* sensitivity of f *is $s(f) = \max_x s(f,x)$.*

Sensitivity of a function indicates how much the function differs (senses differences) between x and its neighbours. Now, instead of distinct edges we consider disjoint blocks $B_i \subseteq [n]$ and their characteristic vertices x_{B_i}.

Definition 2 *The* (local) block sensitivity $bs(f,x)$ *of f on x is defined as a maximum number of* disjoint *blocks $B_1, \ldots, B_k$ such that $f(x) \neq f(x \oplus x_{B_i})$ for every $i \in [k]$. The* block sensitivity of f *is $bs(f) = \max_x bs(f,x)$.*

Similarly to sensitivity, block sensitivity indicates how many subcubes sharing only x see a change of $f(x)$.

Definition 3 *Two complexity measures of Boolean functions A and B are* polynomially related *if there exist polynomials p_1, p_2 over $\mathbb{R}$ such that for every Boolean function f, $A(f) \leq p_1(B(f))$ and $B(f) \leq p_2(A(f))$.*

Are $s(f)$ and $bs(f)$ polynomially related? Obviously $s(f) \leq bs(f)$ for every f. But what about the other direction?

Conjecture 4 (Nisan and Szegedy [4]) *There is an absolute constant $c > 0$ such that for every boolean function f,*

$$bs(f) \leq s(f)^c.$$

Let's take a look at Rubinstein's function [5] defined as follows: For $n = k^2$ where k is even, let $f(x_1, \ldots, x_k, x_{k+1}, \ldots, x_{2k}, \ldots, x_{k^2-k+1}, \ldots, x_{k^2}) = 1$ if and only if at least one block $x_{\ell k+1}, \ldots, x_{\ell(k+1)}$ for $0 \leq \ell < k$ contains exactly two consecutive 1's and the first 1 is at an odd position.

For this function, $bs(f) = n/2$, which is attained for $x = 0^n$ and $B_i = \{2i-1, 2i\}$. However, $s(f) = k = \sqrt{n}$. Therefore, if Conjecture 4 holds, then c needs to be at least 2.

2 Other complexity measures of Boolean functions

For a boolean function f we may consider different ways of measuring its complexity. Then, we can ask how are these related and specifically, whether they are related polynomially.

Any Boolean function f can be computed using a binary decision tree where internal vertices represent a bit query, left and right children represent the bit being 0 or 1 and leaves encode a decision for the final value.

Definition 5 *A* decision tree complexity $D(f)$ *is the minimal depth of a decision tree computing* f*.*

Let $x, y \in \{0,1\}^n$ and $S \subseteq [n]$. Then with $x_S = y_S$ we denote that $x_i = y_i$ for all $i \in S$.

Definition 6 *A* certificate *of* f *on* x *is* $S \subseteq [n]$ *such that for all* y*, if* $x_S = y_S$*, then* $f(x) = f(y)$*. The* (local) certificate complexity $C(f, x)$ *is a minimal size of a certificate on* x*. The* certificate complexity *is* $C(f) = \max_{x \in \{0,1\}^n} C(f, x)$*.*

A certificate represents a subcube containing x where f is constant. The certificate complexity then measures the maximum dimension of such a subcube.

For every Boolean function $f : \{-1,1\}^n \to \{-1,1\}$ there is a *unique* multilinear polynomial $p : \mathbb{R}^n \to \mathbb{R}$ representing f. Specifically,

$$f = \sum_{S \subseteq [n]} \hat{f}(s) \chi_S \qquad \text{where } \chi_S(x) = \prod_{i \in S} x_i, \chi_\emptyset(x) = 1.$$

This representation is called a Fourier transform and $\hat{f}(S)$ is a Fourier coefficient at S.

Definition 7 *The* degree $\deg(f)$ *of* f *is the degree of the polynomial* p *representing* f*; that is,* $\max |S|$ *with* $\hat{f}(S) \neq 0$*.*

There are other measures of complexity, such as bounded-error randomised decision tree complexity, quantum decision tree complexity with bounded error or approximate degree.

It is known that all the above measures except sensitivity are polynomially related. In particular, it is known [6] that $bs(f) \leq \deg(f)^2$ for every f.

3 Induced subgraphs and sensitivity

Now, we connect sensitivity and properties of induced subgraphs of hypercubes.

Definition 8 *For an induced subgraph* H *of* Q_n*, let* $\Gamma(H) = \max\{\Delta(H), \Delta(Q_n - H)\}$*.*

Theorem 9 (Gotsman and Linial [2]) *The following statements are equivalent for any monotone function* $h : \mathbb{N} \to \mathbb{R}$*:*

(a) For any induced subgraph H *of* Q_n *with* $|V(H)| \neq 2^{n-1}$*, we have* $\Gamma(H) \geq h(n)$*.*

(b) For any Boolean function f, we have $s(f) \geq h(\deg(f))$.

Proof We will use an altered claim (b') and prove its equivalence with (b):

(b') For any Boolean function f with $\deg(f) = n$, we have $s(f) \geq h(n)$.

Trivially, (b) implies (b'). For the other direction, let $\deg(f) = d$ and without loss of generality assume it has a nonzero coefficient at the monomial $x_1 \dots x_d$. Define $g(x_1, \dots, x_d) = f(x_1, \dots, x_d, 0, \dots, 0)$. Then $\deg(g) = d$ and $s(f) \geq s(g) \geq h(d)$, where the first inequality holds by definition and second due to (b').

Now we show that the (a) implies (b'). Suppose there is a Boolean function f with $\deg(f) = n$ and $s(f) < h(n)$. Consider a function $g(x) = f(x) \cdot p(x)$ where $p(x) = \prod_{i \in [n]} x_i$ is the parity function. Let H be the subgraph of Q_n induced on true points of g.

Then, $\hat{g}(S) = \hat{f}(\bar{S})$ for any $S \subseteq [n]$. In particular, $\hat{g}(\emptyset) = \hat{f}([n]) \neq 0$, since $\deg(f) = n$. Thus, $\mathbb{E}_x[g(x)] = \hat{g}(\emptyset) \neq 0$, since the values of $\hat{g}(S)$ cancel out for $S \neq \emptyset$. Therefore, $|V(H)| \neq 2^{n-1}$. Furthermore, $\deg_H(x) = n - s(g, x) = s(f, x)$ for every $x \in V(H)$ since g swaps 0 and 1. Similarly, $\deg_{Q_n - H}(x) = s(f, x)$ for every $x \notin V(H)$. Therefore, $\Gamma(H) \leq s(f) < h(n)$, contradicting the premise.

Finally, to show that (b') implies (a), observe that all steps in the above proof of the reverse implication are reversible. ■

There exists H such that $|V(H)| = 2^{n-1}$, for which $\Gamma(H) = 0$; take only the even vertices (having even number of 1's). A lower bound function h on a symmetric degree of "unbalanced" induced subgraphs gives a lower bound on sensitivity of any Boolean function in terms of its degree. How large $h : \mathbb{N} \to \mathbb{R}$ could be so that (a) equivalent with (b) is true?

- Rubenstein's function f for $n = k^2$ and k even has $s(f) = \sqrt{n}$ [5]. It can be shown that $\hat{f}([n]) = (-1)^k (k-1)^k / 2^{k^2} \neq 0$, so $\deg(f) = n$. Therefore, $h(n) \leq \sqrt{n}$.
- AND-of-OR's function, for $n = k^2$, defined for k blocks of k variables [1], is $f(x_{1,1}, \dots, x_{k,k}) = \bigwedge_{i=1}^{k} \bigvee_{j=1}^{k} x_{i,j}$. Consider $g(x) = f(x) \cdot p(x)$ and its associated graph H. Then $|V(H)| = 2^{n-1} \pm 1$, depending on parity of k, and $\Gamma(H) = k = \sqrt{n}$. Therefore, $h(n) \leq \sqrt{n}$.

4 Main theorem

Theorem 10 (Huang [3]) *Let H be an arbitrary $(2^{n-1}+1)$-vertex induced subgraph H of Q_n, $n \geq 1$. Then*

$$\Delta(H) \geq \sqrt{n}.$$

Moreover this inequality is tight when n is a perfect square.

Thus, we can take $h(n) = \sqrt{n}$ in (a) as the maximal degree is monotone with respect to subgraphs and we obtain from (b) that:

Corollary 11 *For every boolean function f,*

$$s(f) \geq \sqrt{\deg(f)}.$$

Furthermore, since $bs(f) \leq \deg^2(f)$ [6], we obtain the following:

Corollary 12 *For every Boolean function f,*

$$bs(f) \leq s(f)^4.$$

To prove Theorem 10, we first present a few lemmas:

Lemma 13 (Cauchy's Interlace Theorem) *Let A be a symmetric $n \times n$ matrix and B be a $m \times m$ principal submatrix of A for some $m < n$. If A has eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and B has eigenvalues $\mu_1 \geq \mu_2 \geq \ldots \mu_m$, then for all $1 \leq i \leq m$;*

$$\lambda_i \geq \mu_i \leq \lambda_{i+n-m}.$$

A submatrix of A is *principal* if it is obtained by deleting the same set of rows and columns from A. A principal submatrix of an adjacency matrix is an adjacency matrix of an induced subgraph.

Lemma 14 *For a sequence of symmetric square matrices given by*

$$A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad A_n = \begin{bmatrix} A_{n-1} & I \\ I & -A_{n-1} \end{bmatrix},$$

A_n has eigenvalues $\sqrt{n}$ and $-\sqrt{n}$, both of multiplicity 2^{n-1}.

Proof By induction, $A_n^2 = nI$. For $n = 1$, $A_1^2 = I$. Assuming $A_{n-1}^2 = (n-1)I$,

$$A_n^2 = \begin{bmatrix} A_{n-1}^2 + I & 0 \\ 0 & A_{n-1}^2 + 1 \end{bmatrix} = nI.$$

Therefore, the eigenvalues are either $\sqrt{n}$ or $-\sqrt{n}$. Since $\mathrm{Tr}[A_n] = 0$, exactly half of them are $\sqrt{n}$ and the other half are $-\sqrt{n}$. ■

The idea behind A_n is that it is an adjacency matrix of Q_n up to "suffix parity" of edges, to keep eigenvalues tight to interlacing. By suffix parity we mean that an edge $u * v$ where $uv \in \{0,1\}^{n-1}$ has the parity of its "suffix" v.

Lemma 15 *Suppose H is an m-vertex undirected graph, A is a symmetric matrix with rows and columns indexed by $V(H)$ having entries in $\{-1, 0, 1\}$ so that $A_{uv} = 0$ whenever $uv \notin E(H)$. Let λ_1 be the largest eigenvalue of A. Then,*

$$\Delta(H) \geq \lambda_1.$$

Proof Let v be the eigenvector corresponding to λ_1, that is $Av = \lambda_1 v$. Without loss of generality assume v_1 has the largest absolute value. Then

$$|\lambda_1 v_1| = |(Av)_1| = \Big|\sum_{j=1}^{m} A_{1j} v_j\Big| \leq \sum_{j=1}^{m} |A_{1j}||v_1| \leq \Delta(H)|v_1|.$$

Therefore, $\Delta(H) \geq |\lambda_1|$. ∎

With the lemmas, we are ready to prove the main theorem.

Proof of Theorem 10: Notice that the matrix $|A_n|$, constructed from A_n by taking the absolute value of each entry, is the adjancency matrix of Q_n. The iterative construction of $|A_n|$ is equivalent to taking two copies of Q_{n-1} and then adding the edges in the "new" direction. Additionally, all entries of A_n are in $\{-1, 0, 1\}$.

Let H be a $(2^{n-1}+1)$-vertex induced subgraph of Q_n and A_H be the principal submatrix of A_n induced by H. Then $|A_H|$ is an adjancency matrix of H and A_H has all its entries in $\{-1, 0, 1\}$, therefore A_H satisfies conditions of Lemma 15. Thus,

$$\Delta(H) \geq \lambda_1(A_H).$$

Since A_H is a principal submatrix of A_n of size $(2^{n-1}+1) \times (2^{n-1}+1)$, by Lemma 13,

$$\lambda_1(A_H) \geq \lambda_{2^{n-1}}(A_n).$$

Finally, by Lemma 14, $\lambda_{2^{n-1}}(A_n) = \sqrt{n}$. ∎

Remark

- The best known separation between $bs(f)$ and $s(f)$ is $bs(f) = \frac{2}{3}s(f)^2 - \frac{1}{3}s(f)$ for some f. Theorem 10 only proves that $bs(f) = O(s(f)^4)$.
- What is the minimum number $g(n, k)$ of vertices that always induce in Q_n a subgraph H with $\Delta(H) \geq k$? We have shown that $g(n, \sqrt{n}) = 2^{n-1} + 1$.
- Can we use the idea of suffix parity for other hypercube problems?

References

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