Hypercube problems

Lecture 22

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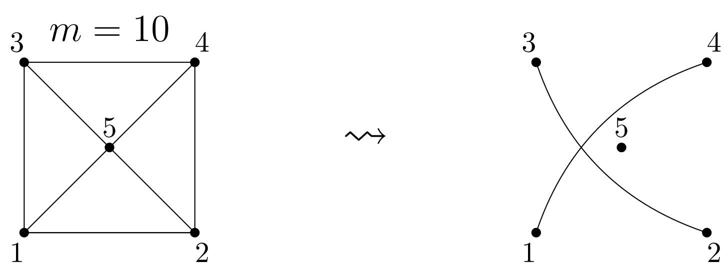
1 Distance magic labelings

The lecture focuses on different *labelings* – bijections between the vertex set of some graph and the set $\{1, 2, \ldots, |V(G)|\}$ with some required properties.

Definition 1 *A bijection $f : V(G) \to \{1, 2, \ldots, |V(G)|\}$ is a* distance magic labeling *of a graph G if there exists some $m \in \mathbb{N}$ such that for every $v \in V(G)$, we have $\sum_{u \in N(v)} f(u) = m$. The number m is called the* magic constant*.*

This labeling is also known as Σ-labeling, 1-vertex-magic-labeling, or neighborhood magic labeling.

Example 1 *An example of a distance magic labeling on a 5-vertex graph with the magic constant $m = 10$ and the distance antimagic labeling on the complement.*



Definition 2 *A* distance antimagic labeling *of a graph G is a bijection $f : V(G) \to \{1, 2, \ldots, |V(G)|\}$ such that for every distinct $v, w \in V(G)$, we have $\sum_{u \in N(v)} f(u) \neq \sum_{u \in N(w)} f(u)$.*

The concept of magic labelings is motivated by its close connection with magic squares and rectangles, and can be used in the design of "fair" incomplete tournaments.

Example 2 *Assume that we have five players Alice, Bob, Carl, Diana, Edith that are ranked from 1 to 5 best-to-worst in a two-player game. Then, if they play a full tournament (i.e., each pair of players plays exactly once), the sum of ranks of the opponents of Alice, the best player, is the largest, while Edith, who is the worst player, has the smallest sum of opponents' ranks.*

In a way, this can be seen an unequal as the tournament will be more demanding to play for Edith than for Alice. Therefore, we could try to create an "equal" tournament, where *the sum of the opponents' ranks each player plays a game against is the same. (Note that we do not expect each player to play the same number of games!) Formally, we are looking for some subgraph of K_5 that has a distance magic labeling.*

One such labeling can be seen in the previous example. Moreover, its complement yields an antimagic labeling of a subgraph of K_5 which corresponds to a "fair" incomplete tournament. The fairness here refers to the fact that the differences between the sums of opponents ranks are the same as in the original complete tournament.

The following table summarizes our observations.

Name	*Ranking*	*Sum of opponents' ranks*		
		complete	*incomplete equal-strength*	*incomplete "fair"*
Alice	*1*	14	m	$14 - m$
Bob	*2*	13	m	$13 - m$
Carl	*3*	12	m	$12 - m$
Diana	*4*	11	m	$11 - m$
Edith	*5*	10	m	$10 - m$

1.1 Uniqueness of the magic constant

A natural question is to ask whether a graph can have multiple magic constants. However, that is not the case and we show that given a graph G, if the magic constant m exists, it is unique.

Definition 3 *Let G be a graph. A function $f : V(G) \to [0,1]$ is a* fractional total domination function *in G if for every $v \in V(G)$, we have $\sum_{u \in N(u)} f(v) \geq 1$. We also define the* size of f *as $|f| := \sum_{v \in V(G)} f(v)$. The* fractional total domination number *of G is $\gamma_{f_t}(G) = \min_{f \text{ f.t.d. function}} |f|$.*

Theorem 4 ([8]) *If G admits distance magic labeling with a magic constant m, then $m = \frac{n(n+1)}{2\gamma_{f_t}(G)}$, where $n = |V(G)|$.*

Proof Let A be the adjacency matrix of the graph G, f its distance magic labeling with a magic constant m, and g a fractional total domination function with $|g| = \gamma_{f_t}(G)$. We take the vectors $\mathbf{f} = (f(v_1), \ldots, f(v_n)), \mathbf{g} = (g(v_1), \ldots, g(v_n))$, $\mathbf{1} = (1, \ldots, 1)^\mathsf{T} \in \mathbb{R}^n$. Then, we notice that $A\mathbf{f}^\mathsf{T} = m\mathbf{1}$ (as for each vector, we simply calculate the sum of the labels of its neighbors) and $A\mathbf{g}^\mathsf{T} \geq \mathbf{1}$ as otherwise, g would not fulfill the "domination" part of its definition for some vertex.
Then, we notice

$$\begin{aligned}
\mathbf{f}A\mathbf{g}^\mathsf{T} &= (\mathbf{f}A\mathbf{g}^\mathsf{T})^\mathsf{T}, \text{ as } \mathbf{f}A\mathbf{g}^\mathsf{T} \text{ is a } 1 \times 1 \text{ matrix} \\
&= \mathbf{g}A\mathbf{f}^\mathsf{T} \\
&= \mathbf{g}m\mathbf{1}, \text{ by the first observed equality} \\
&= m \cdot |g| \\
&= m\gamma_{f_t}(G)
\end{aligned}$$

Also,

$$\begin{aligned}\mathbf{f}A\mathbf{g}^{\mathsf{T}} &= f \cdot (\ell_1, \dots, \ell_n)^{\mathsf{T}} \\ &= \sum_{i=1}^{n} f(v_i)\ell_i \\ &\geq \sum_{i=1}^{n} f(v_i), \text{ as } \ell_i \geq 1 \text{ by the second observed inequality} \\ &= \frac{n(n+1)}{2}\end{aligned}$$

and therefore, we have $m \geq \frac{n(n+1)}{2\gamma_{f_t}(G)}$.

On the other hand, we can take $h : V(G) \to [0,1]$ with $h(v) = f(v)/m$. Such h is clearly a fractional total domination function and $\gamma_{f_t}(G) \leq |h| = |f|/m = \frac{n(n+1)}{2m}$ which yields $m \leq \frac{n(n+1)}{2\gamma_{f_t}(G)}$. ■

1.2 Necessary conditions for the existence of a distance magic labeling

However, we would also like to show the inexistence of a distance magic labeling. One of the possible approaches is via the eigenvalues of adjacency matrices – this is even more general, as we can also extend the approach for a different class of labelings.

Definition 5 *A bijection $f : V(G) \to \{1, 2, \dots, |V(G)|\}$ is a* closed distance magic labeling *of a graph G if there exists some $m \in \mathbb{N}$ such that for every $v \in V(G)$, we have $\sum_{u \in N[v]} f(u) = m$. (Note that the only difference is that we sum over the closed neighborhood of v.)*

Observation 6 ([1, 10]) *If G is a regular graph with a distance magic labeling, then 0 is an eigenvalue of the graph's adjacency matrix A_G. Similarly, if G is a regular graph with a closed distance magic labeling, then -1 is an eigenvalue of A_G.*

Proof For a distance magic labeling f, we have $A\mathbf{f}^{\mathsf{T}} = m\mathbf{1}$ and also $A\mathbf{g}^{\mathsf{T}} = m\mathbf{1}$ for $g = (m/r, \dots, m/r) \in \mathbb{Q}^n$ and the degree r of vertices in G. Then, $A(\mathbf{f}^{\mathsf{T}} - \mathbf{g}^{\mathsf{T}}) = \mathbf{0}$, and therefore 0 is an eigenvalue of A.

Similarly, for a closed distance magic labeling f' with magic number m', $(A+I)\mathbf{f'}^{\mathsf{T}} = m'\mathbf{1}$, $(A+I)\mathbf{g'}^{\mathsf{T}} = m'\mathbf{1}$, where $\mathbf{g'} = (\frac{m'}{r+1}, \dots, \frac{m'}{r+1})$ and $(A+I)(\mathbf{f'}^{\mathsf{T}} - \mathbf{g'}^{\mathsf{T}}) = \mathbf{0}$, and therefore 0 is an eigenvalue of $A + I$, hence -1 is an eigenvalue of A. ■

2 Spectrum of the hypercube

Given the previous necessary condition, we would like to apply it to the hypercube. Therefore, we study the spectrum of the hypercube.

We recall that the set $\{\chi_S : S \subseteq [n]\}$, where $\chi_S(x) := \prod_{i \in S} x_i$ is the *character* of S (note that $\chi_\emptyset(x) = 1$), is an orthogonal basis of the space Ω_n of functions $\{-1,1\}^n \to \mathbb{R}$.

Proposition 7 *The hypercube Q_n has eigenvalues $n-2i$ with multiplicity $\binom{n}{i}$ for $i = 0,\dots,n$.*

Proof Let A be the adjacency matrix of Q_n and consider χ_S as a column vector. Then, $(A\chi_S)(x) = \sum_{i=1}^{n} \chi_S(x \oplus e_i) = (n-2|S|)\chi_S(x)$, as

$$\chi_S(x \oplus e_i) = \begin{cases} -\chi_S(x) & \text{if } i \in S \\ \chi_S(x) & \text{if } i \notin S \end{cases}$$

Therefore, χ_S are the eigenvectors of A and since they are independent, they are the maximum 2^n eigenvectors of A with the corresponding eigenvalues $n-2|S|$. ■

3 Distance magic labelings of hypercubes

The values in the spectrum of the hypercube together with the necessary conditions yield the following corollary.

Corollary 8 *Q_n has no distance magic labeling if n is odd. Similarly, Q_n has no closed distance magic labeling if n is even.*

However, it turns out that hypercubes do not have a distance magic labeling for any even dimension either.

Proposition 9 ([4]) *Q_n has no distance magic labeling if $n \equiv 0 \mod 4$.*

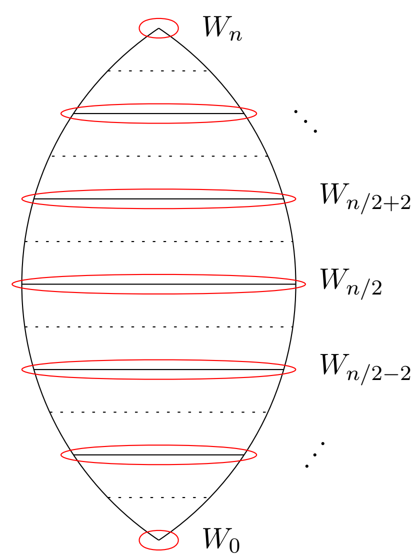
Proof

For contradiction, we suppose that Q_n has a distance magic labeling f and for all $i \in \{0,1,\dots,n\}$, we define $W_i = \sum_{u\in\{0,1\}^n,\, |u|=i} f(u)$. Then, for every $0 < i < n$, we have

$$\binom{n}{i} m = (n-i+1)W_{i-1} + (i+1)W_{i+1}, \qquad (1)$$

where m is the magic constant. This follows easily as if we take a look at all $\binom{n}{i}$ vertices in the level i, the sum of labels on their neighbors is always m. Counting in a different way, each vertex in level $i-1$ is adjacent to $n-i+1$ vertices in the level i and each vertex in the level $i+1$ is adjacent to $i+1$ vertices in the level i, which shows that the two values are the same.

By the automorphism that takes the complements of all bits in each vertex, we claim that $W_{n/2-2i} = W_{n/2+2i}$ for $i = 0,\dots,n/4$. This can be shown by induction on i: the base case is trivial as then we simply have $W_{n/2} = W_{n/2}$ and for the induction step, we can easily observe that the claimed equality also holds by the observed equality (1) for odd levels (dotted in the illustration), properties of the binomial coefficient and the fact that $n/2-2i = n-(n/2+2i)$.



In particular, the claim implies that $f(0^n) = W_0 = W_n = f(1^n)$, which is a contradiction with injectivity of f. ■

The argument in the proof can be generalized to a broader class of graphs. We say that a connected graph G of diameter d is *distance regular* if there exist integers $b_i, c_i : 0 \le i \le d$ such that for any two vertices x, y with $\mathrm{dist}(x, y) = i$, there are exactly b_i neighbors of x at distance $i+1$ to y and exactly c_i neighbors of x at distance $i-1$ to y. Moreover, if every vertex also has a unique vertex at distance d, we call G an *antipodal double cover*.

In fact, we can even ensure some stronger properties for distance magic labelings on antipodal double covers.

Definition 10 *For a graph G and $D \subseteq \{0, 1, \ldots, |V(G)|\}$, we define the D-*distance neighborhood *$N_D(u) = \{v \in V(G) \colon d(u, v) \in D\}$. A bijection $f : V(G) \to \{1, \ldots, |V(G)|\}$ is a D-distance magic labeling if there is a (magic) constant m such that for every $u \in V$: $\sum_{v \in N_D(u)} f(v) = m$.*

Theorem 11 ([3]) *If f is a distance magic labeling (resp. closed distance magic labeling) of an antipodal double cover G with diameter d, then f is an $\{i, d-i\}$-distance magic labeling for every i. Moreover, if G is also bipartite and f is a distance magic labeling, then $d \equiv 2 \mod 4$.*

3.1 Neighbor-balanced mappings

While working with r-regular graphs, we may use the labels $0, \ldots, |V(G)| - 1$ with $m = \frac{r(|V|-1)}{2}$. Moreover, in hypercubes, we work with the labels in binary representation.

Neighbor-balanced mappings are then as a useful concept that is easier to work with, however they are more restricted than general distance magic labelings.

Definition 12 *A set $A \subseteq \mathbb{Z}_2^n$ is* balanced *if $\forall i \in [n] : |\{u \in A \colon u_i = 0\}| = |\{u \in A \colon u_i = 1\}|$. A bijection $f : \mathbb{Z}_2^n \to \mathbb{Z}_2^n$ is* neighbor-balanced *if for each $u \in \mathbb{Z}_2^n$, the set $\{f(u \oplus e_i) \colon i \in [n]\}$ is balanced.*

We observe that any neighbor-balanced function on $\mathbb{Z}_2^n$ is a distance magic labeling of Q_n.

Theorem 13 ([5]) *For every $n \equiv 2 \mod 4$, there exists a neighbor-balanced function $f : \mathbb{Z}_2^n \to \mathbb{Z}_2^n$.*

Proof We assume that there exists a regular $n \times n$ matrix M over $\mathbb{Z}_2$ with a balanced set of columns $C = \{c_1, \ldots, c_n\}$. Then, we define $f(u) = Mu^\mathsf{T}$. By the regularity of M, we have that f is injective. Furthermore, for every vertex u, we have

$$\{f(u \oplus e_i)\}_{i \in [n]} \overset{\text{linearity}}{=} \{f(u) \oplus f(e_i)\}_{i \in [n]} = \{f(u) \oplus c_i\}_{i \in [n]} = C \oplus f(u),$$

which is a translation of a balanced set and is therefore balanced as well.

For $n = 4p + 2, p \in \mathbb{N}$, we can take the matrix M in a $(2p, 2, 2p) \times (2p, 2, 2p)$ block representation to be

$$M = \begin{pmatrix} I & 0 & 1 \\ 0 & I & 1 \\ 1 & 0 & I \end{pmatrix},$$

where I is the identity matrix, $\mathbb{1}$ is the unity (all-ones) matrix and $\mathbb{0}$ is the null matrix. ■

Corollary 14 *The hypercube Q_n has a distance magic labeling if and only if $n \equiv 2 \mod 4$.*

We remark that there exist distance magic labelings of the hypercube that are not neighbor-balanced [9], and there are neighbor-balanced functions that are not affine linear [7].

4 Generalized distance magic labelings

We now continue the study of D-distance magic labelings with less constrained set D. We use A_D to denote the $n \times \sum_{d \in D} \binom{n}{d}$ matrix that contains all binary vectors of size $d \in D$ in its columns. The proof of the following lemma is left as an exercise.

Lemma 15 *If $M \in \mathbb{Z}_2^{n \times n}$ is a matrix with a balanced column set and $D \subseteq \{1, 3, 5, \ldots, n-1\}$, then MA_D is balanced as well.*

Theorem 16 ([5]) *If $n \equiv 2 \mod 4$ and $D \subseteq \{1, 3, 5, \ldots, n-1\}$, then Q_n has a D-distance magic labeling.*

Proof We use the same (balanced) M as in the proof of existence of a neighbor-balanced functions and we set $\{c_1, \ldots, c_\ell\}$ to be the columns of A_D. Then $\{f(u \oplus c_i) \colon i \in [\ell]\} = \{f(u) + f(c_i) \colon i \in [\ell]\} = \{Mc_i \colon i \in [\ell]\} \oplus f(u)$, which is a translation of a balanced set of columns of MA_D and hence is balanced. ■

The following two results are mentioned without proof.

Theorem 17 ([3]) *If $n \equiv 2 \mod 4$ and $D = E \cup \bigcup_{i \in I} \{i, n-i\}$, where $E \subseteq \{1, 3, 5, \ldots, n-1\}$ is nonempty and $I \subseteq \{0, 1, \ldots, n/2\}$ with $E, \bigcup_{i \in I} \{i, n-i\}$ disjoint, then Q_n has a D-distance magic labeling.*

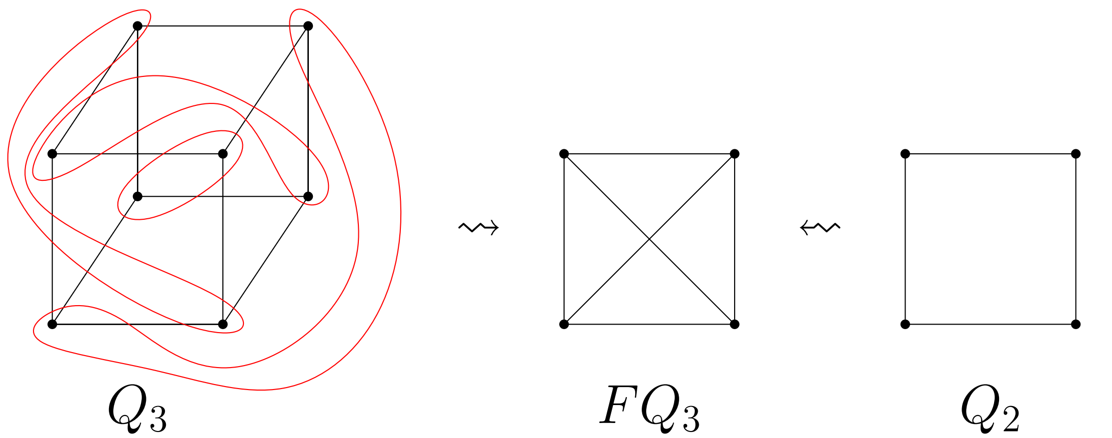
Problem 1 *Are the sets D is the form from the preceding theorem the only sets D such that Q_n with $n \equiv 2 \mod 4$ has a D-distance magic labeling?*

Theorem 18 ([3]) *The hypercube Q_n has a (closed) $\{0, 1\}$-distance magic labeling if and only if $n \equiv 1 \mod 4$.*

5 Distance magic labelings of folded cubes

A *folded cube* FQ_n of dimension n is the graph obtained from Q_n by gluing together every pair of antipodal vertices. Equivalently, we can obtain FQ_n by adding edges between every pair of antipodal vertices in Q_{n-1}.

Example 3



Observation 19 *By deleting the first and the last columns and rows of M, we obtain a $4p \times 4p$ matrix M' with balanced set of columns and rank $4p-1$. The labeling $f'(u) = M'u^{\mathsf{T}}$ then uses every even label exactly twice, on the pairs on antipodal vertices.*

Theorem 20 ([6, 11]) *FQ_n has a distance magic labeling if and only if $n \equiv 0 \mod 4$.*

Proof To show this is sufficient, we use $f'/2$: take f' and delete the last bit. For necessity, we use that the spectrum of FQ_n is $\{n-i\colon 0 \le i \le n/2\}$ and Observation 6. ■

We can also show in a similar way that FQ_n has a closed distance magic labeling if and only if $n \equiv 3 \mod 4$.

Note

For further related results, on group distance magic labelings of hypercubes, and on distance magic labelings of Hamming graphs we refer to [2] and [6], respectively.

References

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