Propositional and Predicate Logic - II

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Semantic notions

A proposition φ over $\mathbb{P}$ is

- *is true in* (*satisfied by*) *an assignment* $v\colon \mathbb{P} \to \{0,1\}$, if $\overline{v}(\varphi) = 1$.
  Then v is a *satisfying assignment* for φ, denoted by $v \models \varphi$.
- *valid* (*a tautology*), if $\overline{v}(\varphi) = 1$ for every $v\colon \mathbb{P} \to \{0,1\}$,
  i.e. φ is satisfied by every assignment, denoted by $\models \varphi$.
- *unsatisfiable* (*a contradiction*), if $\overline{v}(\varphi) = 0$ for every $v\colon \mathbb{P} \to \{0,1\}$, i.e.
  $\neg\varphi$ is valid.
- *independent* (*a contingency*), if $\overline{v_1}(\varphi) = 0$ and $\overline{v_2}(\varphi) = 1$ for some
  $v_1, v_2\colon \mathbb{P} \to \{0,1\}$, i.e. φ is neither a tautology nor a contradiction.
- *satisfiable*, if $\overline{v}(\varphi) = 1$ for some $v\colon \mathbb{P} \to \{0,1\}$, i.e. φ is not a contradiction.

Propositions φ and ψ are (logically) *equivalent*, denoted by $\varphi \sim \psi$, if
$\overline{v}(\varphi) = \overline{v}(\psi)$ for every $v\colon \mathbb{P} \to \{0,1\}$, i.e. the proposition $\varphi \leftrightarrow \psi$ is valid.

Models

We reformulate these semantic notions in the terminology of models.

A *model of a language* $\mathbb{P}$ is a truth assignment of $\mathbb{P}$. The class of all models of $\mathbb{P}$ is denoted by $M(\mathbb{P})$. A proposition φ over $\mathbb{P}$ is

- *true in a model* $v \in M(\mathbb{P})$, if $\overline{v}(\varphi) = 1$. Then v is a *model of* φ, denoted by $v \models \varphi$ and $M^{\mathbb{P}}(\varphi) = \{v \in M(\mathbb{P}) \mid v \models \varphi\}$ is the *class of all models* of φ.
- *valid* (*a tautology*) if it is true in every model of the language, denoted by $\models \varphi$.
- *unsatisfiable* (*a contradiction*) if it does not have a model.
- *independent* (*a contingency*) if it is true in some model and false in other.
- *satisfiable* if it has a model.

Propositions φ and ψ are (logically) *equivalent*, denoted by $\varphi \sim \psi$, if they have same models.

Theory

Informally, a theory is a description of “world” to which we restrict ourselves.

- A propositional *theory* over the language $\mathbb{P}$ is any set T of propositions from $\mathrm{PF}_{\mathbb{P}}$. We say that propositions of T are *axioms* of the theory T.
- A *model of theory* T over $\mathbb{P}$ is an assignment $v \in M(\mathbb{P})$ (i.e. a model of the language) in which all axioms of T are true, denoted by $v \models T$.
- A *class of models* of T is $M^{\mathbb{P}}(T) = \{v \in M(\mathbb{P}) \mid v \models \varphi$ for every $\varphi \in T\}$.

  For example, for $T = \{p,\ \neg p \vee \neg q,\ q \rightarrow r\}$ over $\mathbb{P} = \{p, q, r\}$ we have

$$M^{\mathbb{P}}(T) = \{(1,0,0),(1,0,1)\}$$

- If a theory is finite, it can be replaced by a *conjunction* of its axioms.
- We write $M(T,\varphi)$ as a shortcut for $M(T \cup \{\varphi\})$.

Semantics with respect to a theory

Semantic notions can be defined with respect to a theory, more precisely, with respect to its models. Let T be a theory over $\mathbb{P}$. A proposition φ over $\mathbb{P}$ is

- *valid in T* (*true in T*) if it is true in every model of T, denoted by $T \models \varphi$, We also say that φ is a (semantic) *consequence* of T.
- *unsatisfiable* (*contradictory*) *in T* (*inconsistent with T*) if it is false in every model of T,
- *independent (or contingency) in T* if it is true in some model of T and false in some other,
- *satisfiable in T* (*consistent with T*) if it is true in some model of T.

Propositions φ and ψ are *equivalent in T* (*T-equivalent*), denoted by $\varphi \sim_T \psi$, if for every model v of T, $v \models \varphi$ if and only if $v \models \psi$.

Note If all axioms of a theory T are valid (tautologies), e.g. for $T = \emptyset$, then all notions with respect to T correspond to the same notions in (pure) logic.

Universality

The language of propositional logic has *basic* connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$.
In general, we can introduce n-ary connective for any Boolean function, e.g.

$p \downarrow q$	*"neither p nor q"*	(NOR, Peirce arrow)
$p \uparrow q$	*"not both p and q"*	(NAND, Sheffer stroke)

A set of connectives is *universal* if every Boolean function can be expressed as a proposition formed from these connectives.

Proposition $\{\neg , \wedge , \vee\}$ *is universal.*

Proof A function $f\colon \{0,1\}^n \to \{0,1\}$ is expressed by $\bigvee_{v \in f^{-1}[1]} \bigwedge_{i=1}^{n} p_i^{v(i)}$ where $p_i^{v(i)}$ denotes the proposition p_i if $v(i) = 1$; and $\neg p_i$ if $v(i) = 0$.
For $f^{-1}[1] = \emptyset$ we take the proposition $\perp$. $\square$

Proposition $\{\neg , \rightarrow\}$ *is universal.*

Proof $(p \wedge q) \sim \neg(p \rightarrow \neg q)$, $(p \vee q) \sim (\neg p \rightarrow q)$. $\square$

CNF and DNF

- A *literal* is a propositional letter or its negation. Let p^1 be the literal p and let p^0 be the literal $\neg p$. Let $\bar{l}$ denote the *complementary* literal to a literal l.
- A *clause* is a disjunction of literals, by the empty clause we mean $\bot$.
- A proposition is in *conjunctive normal form* (*CNF*) if it is a conjunction of clauses. By the empty proposition in CNF we mean $\top$.
- An *elementary conjunction* is a conjunction of literals, by the empty conjunction we mean $\top$.
- A proposition is in *disjunctive normal form* (*DNF*) if it is a disjunction of elementary conjunctions. By the empty proposition in DNF we mean $\bot$.

Note A clause or an elementary conjunction is both in CNF and DNF.

Observation *A proposition in CNF is valid if and only if each of its clauses contains a pair of complementary literals. A proposition in DNF is satisfiable if and only if at least one of its elementary conjunctions does not contain a pair of complementary literals.*

Transformations by tables

Proposition *Let $K \subseteq \{0,1\}^{\mathbb{P}}$ where $\mathbb{P}$ is finite and $\overline{K} = \{0,1\}^{\mathbb{P}} \setminus K$. Then*

$$M^{\mathbb{P}}\Big(\bigvee_{v \in K} \bigwedge_{p \in \mathbb{P}} p^{v(p)}\Big) = K = M^{\mathbb{P}}\Big(\bigwedge_{v \in \overline{K}} \bigvee_{p \in \mathbb{P}} \overline{p^{v(p)}}\Big)$$

Proof The first equality follows from $w(\bigwedge_{p \in \mathbb{P}} p^{v(p)}) = 1$ if and only if $w = v$. Similarly, the second one follows from $w(\bigvee_{p \in \mathbb{P}} \overline{p^{v(p)}}) = 1$ if and only if $w \neq v$. □

For example, $K = \{(1,0,0),(1,1,0),(0,1,0),(1,1,1)\}$ can be modeled by

$$(p \wedge \neg q \wedge \neg r) \vee (p \wedge q \wedge \neg r) \vee (\neg p \wedge q \wedge \neg r) \vee (p \wedge q \wedge r) \sim$$
$$(p \vee q \vee r) \wedge (p \vee q \vee \neg r) \wedge (p \vee \neg q \vee \neg r) \wedge (\neg p \vee q \vee \neg r)$$

Corollary *Every proposition has CNF and DNF equivalents.*

Proof The value of a proposition φ depends only on the assignment of $\mathrm{var}(\varphi)$ which is finite. Hence we can apply the above proposition for $K = M^{\mathbb{P}}(\varphi)$ and $\mathbb{P} = \mathrm{var}(\varphi)$. □

Transformations by rules

Proposition *Let φ' be the proposition obtained from φ by replacing some occurrences of a subformula ψ with ψ'. If $\psi \sim \psi'$, then $\varphi \sim \varphi'$.*

Proof By induction on the structure of the formula. □

(1) $(\varphi \rightarrow \psi) \sim (\neg\varphi \vee \psi)$, $\quad (\varphi \leftrightarrow \psi) \sim ((\neg\varphi \vee \psi) \wedge (\neg\psi \vee \varphi))$

(2) $\neg\neg\varphi \sim \varphi$, $\quad \neg(\varphi \wedge \psi) \sim (\neg\varphi \vee \neg\psi)$, $\quad \neg(\varphi \vee \psi) \sim (\neg\varphi \wedge \neg\psi)$

(3) $(\varphi \vee (\psi \wedge \chi)) \sim ((\psi \wedge \chi) \vee \varphi) \sim ((\varphi \vee \psi) \wedge (\varphi \vee \chi))$

(3)' $(\varphi \wedge (\psi \vee \chi)) \sim ((\psi \vee \chi) \wedge \varphi) \sim ((\varphi \wedge \psi) \vee (\varphi \wedge \chi))$

Proposition *Every proposition can be transformed into CNF / DNF applying the transformation rules $(1), (2), (3)/(3)'$.*

Proof By induction on the structure of the formula. □

Proposition *Assume that φ contains only $\neg$, $\wedge$, $\vee$ and φ^* is obtained from φ by interchanging $\wedge$ and $\vee$, and by complementing all literals. Then* $\neg\varphi \sim \varphi^*$.

Proof By induction on the structure of the formula. □

Consequence of a theory

The *consequence* of a theory T over $\mathbb{P}$ is the set $\theta^{\mathbb{P}}(T)$ of all propositions that are valid in T, i.e.

$$\theta^{\mathbb{P}}(T) = \{\varphi \in \mathrm{PF}_{\mathbb{P}} \mid T \models \varphi\}.$$

Proposition *For every theories $T \subseteq T'$ and propositions $\varphi, \varphi_1, \ldots, \varphi_n$ over $\mathbb{P}$,*

(1) $T \subseteq \theta^{\mathbb{P}}(T) = \theta^{\mathbb{P}}(\theta^{\mathbb{P}}(T))$,

(2) $T \subseteq T' \;\Rightarrow\; \theta^{\mathbb{P}}(T) \subseteq \theta^{\mathbb{P}}(T')$,

(3) $\varphi \in \theta^{\mathbb{P}}(\{\varphi_1, \ldots, \varphi_n\}) \;\Leftrightarrow\; \models (\varphi_1 \wedge \ldots \wedge \varphi_n) \rightarrow \varphi$.

Proof Easily from definitions, since $T \models \varphi \;\Leftrightarrow\; M(T) \subseteq M(\varphi)$ and

(1) $M(\theta(T)) = M(T)$,

(2) $T \subseteq T' \;\Rightarrow\; M(T') \subseteq M(T)$,

(3) $\models \psi \rightarrow \varphi \;\Leftrightarrow\; M(\psi) \subseteq M(\varphi),\; M(\varphi_1 \wedge \ldots \wedge \varphi_n) = M(\varphi_1, \ldots, \varphi_n)$. $\square$

Properties of theories

A propositional theory T over $\mathbb{P}$ is *(semantically)*

- *inconsistent* (*unsatisfiable*) if $T \models \perp$, otherwise is *consistent* (*satisfiable*),
- *complete* if it is consistent, and $T \models \varphi$ or $T \models \neg\varphi$ for every $\varphi \in \mathrm{PF}_{\mathbb{P}}$, i.e. no proposition over $\mathbb{P}$ is independent in T,
- an *extension* of a theory T' over $\mathbb{P}'$ if $\mathbb{P}' \subseteq \mathbb{P}$ and $\theta^{\mathbb{P}'}(T') \subseteq \theta^{\mathbb{P}}(T)$; we say that an extension T of a theory T' is *simple* if $\mathbb{P} = \mathbb{P}'$; and *conservative* if $\theta^{\mathbb{P}'}(T') = \theta^{\mathbb{P}}(T) \cap \mathrm{VF}_{\mathbb{P}'}$,
- *equivalent* with a theory T' if T is an extension of T' and vice-versa,

Observation *Let T and T' be theories over $\mathbb{P}$. Then T is (semantically)*

(1) *consistent if and only if it has a model,*

(2) *complete if and only if it has a single model,*

(3) *extension of T' if and only if $M^{\mathbb{P}}(T) \subseteq M^{\mathbb{P}}(T')$,*

(4) *equivalent with T' if and only if $M^{\mathbb{P}}(T) = M^{\mathbb{P}}(T')$.*

Algebra of propositions

Let T be a consistent theory over $\mathbb{P}$. On the quotient set $\mathrm{PF}_{\mathbb{P}}/{\sim_T}$ we define operations $\neg$, $\wedge$, $\vee$, $\bot$, $\top$ (correctly) by use of representatives, e.g.

$$[\varphi]_{\sim_T} \wedge [\psi]_{\sim_T} = [\varphi \wedge \psi]_{\sim_T}$$

Then $AV^{\mathbb{P}}(T) = \langle \mathrm{PF}_{\mathbb{P}}/{\sim_T}, \neg, \wedge, \vee, \bot, \top \rangle$ is *algebra of propositions* for T.

Since $\varphi \sim_T \psi \;\Leftrightarrow\; M(T,\varphi) = M(T,\psi)$, it follows that $h([\varphi]_{\sim_T}) = M(T,\varphi)$ is a (well-defined) injective function $h\colon \mathrm{PF}_{\mathbb{P}}/{\sim_T} \to \mathcal{P}(M(T))$ and

$$\begin{aligned} h(\neg[\varphi]_{\sim_T}) &= M(T) \setminus M(T,\varphi) \\ h([\varphi]_{\sim_T} \wedge [\psi]_{\sim_T}) &= M(T,\varphi) \cap M(T,\psi) \\ h([\varphi]_{\sim_T} \vee [\psi]_{\sim_T}) &= M(T,\varphi) \cup M(T,\psi) \\ h([\bot]_{\sim_T}) = \emptyset, \quad & h([\top]_{\sim_T}) = M(T) \end{aligned}$$

Moreover, h is *surjective* if $M(T)$ is *finite*.

Corollary If T is a consistent theory over a finite $\mathbb{P}$, then $AV^{\mathbb{P}}(T)$ is a Boolean algebra *isomorphic* via h to the (finite) algebra of sets $\underline{\mathcal{P}}(M(T))$.

Analysis of theories over finite languages

Let T be a consistent theory over $\mathbb{P}$ where $|\mathbb{P}| = n \in \mathbb{N}^+$ and $m = |M^{\mathbb{P}}(T)|$.
Then the number of (mutually) nonequivalent

- propositions (or theories) over $\mathbb{P}$ is 2^{2^n},
- propositions over $\mathbb{P}$ that are valid (contradictory) in T is $2^{2^n - m}$,
- propositions over $\mathbb{P}$ that are independent in T is $2^{2^n} - 2.2^{2^n - m}$,
- simple extensions of T is 2^m, out of which 1 is inconsistent,
- complete simple extensions of T is m.

And the number of (mutually) T-nonequivalent

- propositions over $\mathbb{P}$ is 2^m,
- propositions over $\mathbb{P}$ that are valid (contradictory) (in T) is 1,
- propositions over $\mathbb{P}$ that are independent (in T) is $2^m - 2$.

Proof By the bijection of $\mathrm{PF}_{\mathbb{P}}/\!\sim$ resp. $\mathrm{PF}_{\mathbb{P}}/\!\sim_T$ with $\mathcal{P}(M(\mathbb{P}))$ resp. $\mathcal{P}(M^{\mathbb{P}}(T))$ it suffices to determine the number of appropriate subsets of models. $\square$