

Propositional and Predicate Logic - IV

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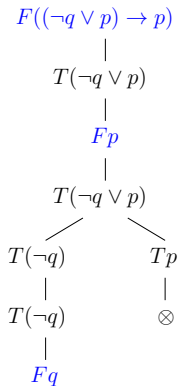
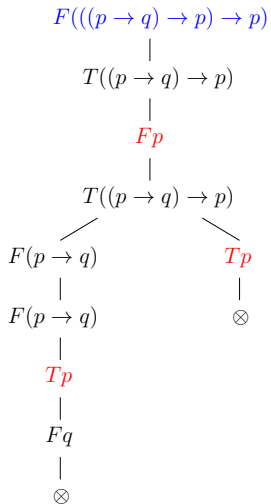
Tableau method - introduction

We assume that the language is fixed and **countable**, i.e. the set \mathbb{P} of propositional letters is countable. Then every **theory** over \mathbb{P} is **countable**.

Main features of the tableau method (*informally*)

- a **tableau** for a formula φ is a binary labeled tree representing systematic search for **counterexample** to φ , i.e. a model of theory in which φ is false,
- a formula is **proved** if every branch in tableau 'fails', i.e. counterexample was not found. In this case the (systematic) tableau will be **finite**,
- if a counterexample exists, there will be a branch in a (finished) tableau that provides us with this counterexample, but this branch can be **infinite**.

Introductory examples



Atomic tableaux

An *atomic tableau* is one of the following trees (labeled by entries), where p is any propositional letter and φ, ψ are any propositions.

Tp	Fp	$T(\varphi \wedge \psi)$ $\quad $ $T\varphi$ $\quad $ $T\psi$	$F(\varphi \wedge \psi)$ $\quad / \quad \backslash$ $F\varphi \quad F\psi$	$T(\varphi \vee \psi)$ $\quad / \quad \backslash$ $T\varphi \quad T\psi$	$F(\varphi \vee \psi)$ $\quad $ $F\varphi$ $\quad $ $F\psi$
$T(\neg\varphi)$ $\quad $ $F\varphi$	$F(\neg\varphi)$ $\quad $ $T\varphi$	$T(\varphi \rightarrow \psi)$ $\quad / \quad \backslash$ $F\varphi \quad T\psi$	$F(\varphi \rightarrow \psi)$ $\quad $ $T\varphi$ $\quad $ $F\psi$	$T(\varphi \leftrightarrow \psi)$ $\quad / \quad \backslash$ $T\varphi \quad F\varphi$ $\quad \quad \quad $ $T\psi \quad F\psi$	$F(\varphi \leftrightarrow \psi)$ $\quad / \quad \backslash$ $T\varphi \quad F\varphi$ $\quad \quad \quad $ $F\psi \quad T\psi$

All tableaux will be formally defined with atomic tableaux and rules how to expand them.

Tableaux

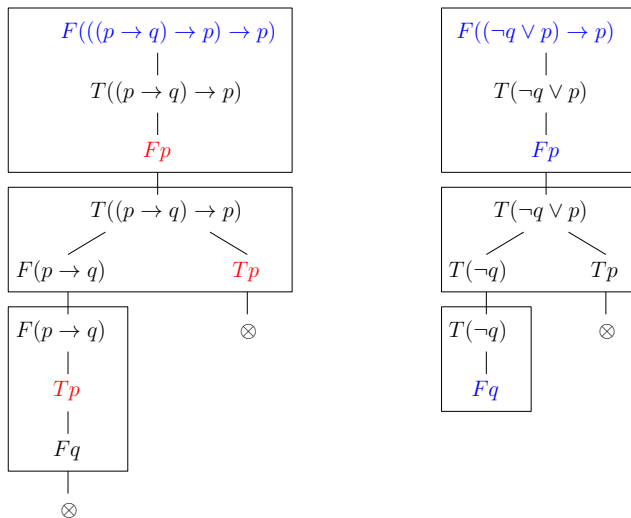
A **finite tableau** is a binary tree labeled with entries described (inductively) by

- (i) every atomic tableau is a finite tableau,
- (ii) if P is an entry on a branch V in a finite tableau τ and τ' is obtained from τ by **appending** the atomic tableaux for P at the **end of branch V** , then τ' is also a finite tableau,
- (iii) every finite tableau is formed by a **finite** number of steps (i), (ii).

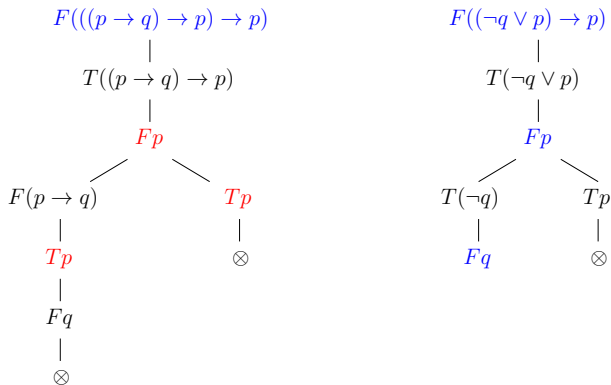
A **tableau** is a sequence $\tau_0, \tau_1, \dots, \tau_n, \dots$ (finite or infinite) of finite tableaux such that τ_{n+1} is formed from τ_n by an application of (ii), formally $\tau = \cup \tau_n$.

Remark *It is not specified how to choose the entry P and the branch V for expansion. This will be specified in **systematic tableaux**.*

Construction of tableaux



Convention



We will not **write** the entry that is expanded again on the branch.

Remark They will actually be needed later in predicate tableau method.

Tableau proofs

Let P be an entry on a branch V in a tableau τ . We say that

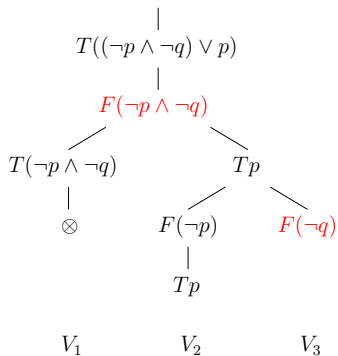
- the entry P is *reduced* on V if it *occurs* on V as a root of an atomic tableau, i.e. it was already expanded on V during the construction of τ ,
- the branch V is *contradictory* if it contains entries $T\varphi$ and $F\varphi$ for some proposition φ , otherwise V is *noncontradictory*. The branch V is *finished* if it is contradictory or every entry on V is already reduced on V ,
- the tableau τ is *finished* if every branch in τ is finished, and τ is *contradictory* if every branch in τ is contradictory.

A *tableau proof* (*proof by tableau*) of φ is a *contradictory tableau* with the root entry $F\varphi$. φ is *(tableau) provable*, denoted by $\vdash \varphi$, if it has a tableau proof.

Similarly, a *refutation* of φ by *tableau* is a *contradictory tableau* with the root entry $T\varphi$. φ is *(tableau) refutable* if it has a refutation by tableau, i.e. $\vdash \neg\varphi$.

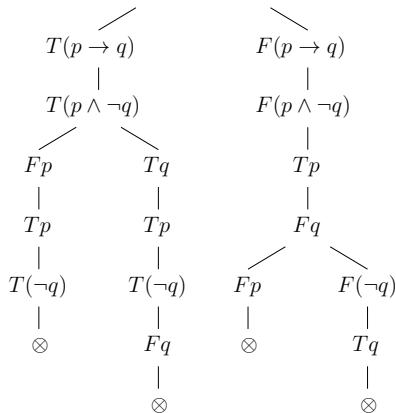
Examples

$$F(((\neg p \wedge \neg q) \vee p) \rightarrow (\neg p \wedge \neg q))$$



a)

$$T((p \rightarrow q) \leftrightarrow (p \wedge \neg q))$$



b)

a) $F(\neg p \wedge \neg q)$ not reduced on V_1 , V_1 contradictory, V_2 finished, V_3 unfinished,

b) a (tableau) refutation of φ : $(p \rightarrow q) \leftrightarrow (p \wedge \neg q)$, i.e. $\vdash \neg\varphi$.

Tableau from a theory

How to add axioms of a given theory into a proof?

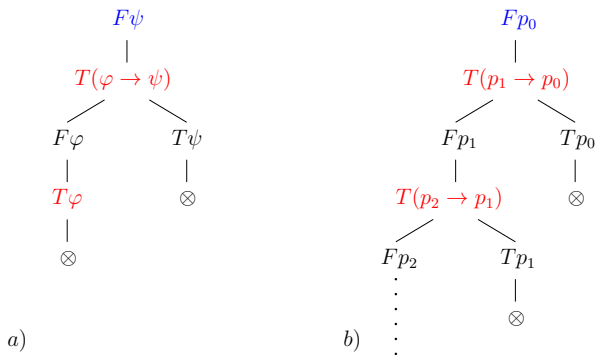
A *finite tableau from a theory* T is generalized tableau with an additional rule (ii)' if V is a branch of a finite tableau (from T) and $\varphi \in T$, then by appending $T\varphi$ at the end of V we obtain a finite tableau from T .

We generalize other definitions by appending “from T ”.

- a *tableau from T* is a sequence $\tau_0, \tau_1, \dots, \tau_n, \dots$ of finite tableaux from T such that τ_{n+1} is formed from τ_n applying (ii) or (ii)', formally $\tau = \cup \tau_n$,
- a *tableau proof of φ from T* is a contradictory tableaux from T with $F\varphi$ in the root. $T \vdash \varphi$ denotes that φ is (*tableau*) *provable from T* .
- a *refutation of φ by a tableau from T* is a contradictory tableau from T with the root entry $T\varphi$.

Unlike in previous definitions, a branch V of a tableau from T is *finished*, if it is contradictory, or every entry on V is already reduced on V and, *moreover*, V contains $T\varphi$ for every $\varphi \in T$.

Examples of tableaux from theories



a) A tableau **proof** of ψ from $T = \{\varphi, \varphi \rightarrow \psi\}$, so $T \vdash \psi$.

b) A **finished** tableau with the root Fp_0 from $T = \{p_{n+1} \rightarrow p_n \mid n \in \mathbb{N}\}$.

All branches are finished, the leftmost branch is **noncontradictory** and infinite. It provides us with the (only one) model of T in which p_0 is false.

Systematic tableaux

We describe a systematic construction that leads to a *finished* tableau.

Let R be an entry and $T = \{\varphi_0, \varphi_1, \dots\}$ be a (possibly infinite) theory.

- (1) We take the atomic tableau for R as τ_0 . Till possible, proceed as follows.
- (2) Let P be the **leftmost** entry in the **smallest** level as possible of the tableau τ_n s.t. P is not reduced on some noncontradictory branch **through** P .
- (3) Let τ'_n be the tableau obtained from τ_n by appending the atomic tableau for P to every noncontradictory branch through P . (If P does not exist, we take $\tau'_n = \tau_n$.)
- (4) Let τ_{n+1} be the tableau obtained from τ'_n by appending $T\varphi_n$ to every noncontradictory branch that does not contain $T\varphi_n$ yet. (If φ_n does not exist, we take $\tau_{n+1} = \tau'_n$.)

The *systematic tableau* from T for the entry R is the result of the above construction, i.e. $\tau = \cup \tau_n$.

Systematic tableau - being finished

Proposition *Every systematic tableau is finished.*

Proof Let $\tau = \cup \tau_n$ be a systematic tableau from $T = \{\varphi_0, \varphi_1, \dots\}$ with root R .

- If a branch is noncontradictory in τ , its **prefix** in every τ_n is noncontradictory as well.
- If an entry P is unreduced on some branch in τ , it is unreduced on its prefix in every τ_n as well (assuming P occurs on this prefix).
- There are only finitely many entries in τ in levels up to the level of P .
- Thus, if P was unreduced on some noncontradictory branch in τ , it would be considered in some step (2) and reduced by step (3).
- By step (4) every $\varphi_n \in T$ will be (no later than) in τ_{n+1} on every noncontradictory branch.
- Hence the systematic tableau τ has all branches finished. \square

Finiteness of proofs

Proposition For every contradictory tableau $\tau = \cup \tau_n$ there is some n such that τ_n is a contradictory **finite** tableau.

- **Proof** Let S be the set of nodes in τ that have no pair of contradictory entries $T\varphi, F\varphi$ amongst their predecessors.
- If S was infinite, then by **König's lemma**, the subtree of τ induced by S would contain an infinite branch, and thus τ would not be contradictory.
- Since S is finite, for some m all nodes of S belong to levels up to m .
- Thus every node in level $m + 1$ has a pair of contradictory entries amongst its predecessors.
- Let n be such that τ_n agrees with τ at least up to the level $m + 1$.
- Then every branch in τ_n is contradictory. \square

Corollary If a systematic tableau (from a theory) is a proof, it is finite.

Proof In its construction, only noncontradictory branches are extended. \square

Soundness

We say that an entry P *agrees* with an assignment ν , if P is $T\varphi$ and $\bar{\nu}(\varphi) = 1$, or if P is $F\varphi$ and $\bar{\nu}(\varphi) = 0$. A branch V *agrees* with ν , if every entry on V agrees with ν .

Lemma *Let ν be a model of a theory T that agrees with the root entry of a tableau $\tau = \cup \tau_n$ from T . Then τ contains a branch that agrees with ν .*

Proof By induction we find a sequence V_0, V_1, \dots so that for every n , V_n is a branch in τ_n agreeing with ν and V_n is contained in V_{n+1} .

- By considering all atomic tableaux we verify that base of induction holds.
- If τ_{n+1} is obtained from τ_n without extending V_n , we put $V_{n+1} = V_n$.
- If τ_{n+1} is obtained from τ_n by appending $T\varphi$ to V_n for some $\varphi \in T$, let V_{n+1} be this branch. Since ν is a model of φ , V_{n+1} agrees with ν .
- Otherwise τ_{n+1} is obtained from τ_n by appending the atomic tableau for some entry P on V_n to the end of V_n . Since P agrees with ν and atomic tableaux are verified, V_n can be extended to V_{n+1} as required. \square

Theorem on soundness

We will show that the tableau method in propositional logic is *sound*.

Theorem For every theory T and proposition φ , if φ is tableau provable from T , then φ is valid in T , i.e. $T \vdash \varphi \Rightarrow T \models \varphi$.

Proof

- Let φ be tableau provable from a theory T , i.e. there is a contradictory tableau τ from T with the root entry $F\varphi$.
- Suppose for a contradiction that φ is not valid in T , i.e. there exists a model v of the theory T in which φ is false (a *counterexample*).
- Since the root entry $F\varphi$ agrees with v , by the previous lemma, there is a branch in the tableau τ that agrees with v .
- But this is impossible, since every branch of τ is contradictory, i.e. it contains a pair of entries $T\psi, F\psi$ for some ψ . \square

Completeness

A noncontradictory branch in a finished tableau gives us a *counterexample*.

Lemma Let V be a *noncontradictory* branch of a *finished* tableau τ .

Then V agrees with the following assignment v .

$$v(p) = \begin{cases} 1 & \text{if } Tp \text{ occurs on } V \\ 0 & \text{otherwise} \end{cases}$$

Proof By induction on the structure of formulas in entries occurring on V .

- For an entry Tp on V , where p is a letter, we have $\bar{v}(p) = 1$ by definition.
- For an entry Fp on V , Tp is not on V since V is noncontradictory, thus $\bar{v}(p) = 0$ by definition of v .
- For an entry $T(\varphi \wedge \psi)$ on V , we have $T\varphi$ and $T\psi$ on V since τ is finished. By induction, we have $\bar{v}(\varphi) = \bar{v}(\psi) = 1$, and thus $\bar{v}(\varphi \wedge \psi) = 1$.
- For an entry $F(\varphi \wedge \psi)$ on V , we have $F\varphi$ or $F\psi$ on V since τ is finished. By induction, we have $\bar{v}(\varphi) = 0$ or $\bar{v}(\psi) = 0$, and thus $\bar{v}(\varphi \wedge \psi) = 0$.
- For other entries similarly as in previous two cases. \square

Theorem on completeness

We will show that the tableau method in propositional logic is **complete**.

Theorem For every theory T and proposition φ , if φ is valid in T , then φ is tableau provable from T , i.e. $T \models \varphi \Rightarrow T \vdash \varphi$.

Proof Let φ be valid in T . We will show that an arbitrary **finished** tableau (e.g. **systematic**) τ from theory T with the root entry $F\varphi$ is **contradictory**.

- If not, let V be some noncontradictory branch in τ .
- By the previous lemma, there exists an assignment ν such that V agrees with ν , in particular in the root entry $F\varphi$, i.e. $\bar{\nu}(\varphi) = 0$.
- Since V is finished, it contains $T\psi$ for every $\psi \in T$.
- Thus ν is a model of theory T (since V agrees with ν).
- But this contradicts the assumption that φ is valid in T .

Hence the tableau τ is a proof of φ from T . \square

Properties of theories

We introduce syntactic variants of previous semantically defined notions.

Let T be a theory over \mathbb{P} . If φ is provable from T , we say that φ is a *theorem* of T . The set of theorems of T is denoted by

$$\text{Thm}^{\mathbb{P}}(T) = \{\varphi \in \text{PF}_{\mathbb{P}} \mid T \vdash \varphi\}.$$

We say that a theory T is

- *inconsistent* if $T \vdash \perp$, otherwise T is *consistent*,
- *complete* if it is consistent and every proposition is provable or refutable from T , i.e. $T \vdash \varphi$ or $T \vdash \neg\varphi$ for every $\varphi \in \text{PF}_{\mathbb{P}}$,
- *extension* of a theory T' over \mathbb{P}' if $\mathbb{P}' \subseteq \mathbb{P}$ and $\text{Thm}^{\mathbb{P}'}(T') \subseteq \text{Thm}^{\mathbb{P}}(T)$; we say that an extension T of a theory T' is *simple* if $\mathbb{P} = \mathbb{P}'$; and *conservative* if $\text{Thm}^{\mathbb{P}'}(T') = \text{Thm}^{\mathbb{P}}(T) \cap \text{VF}_{\mathbb{P}'}$,
- *equivalent* with a theory T' if T is an extension of T' and vice-versa.

Corollaries

From the soundness and completeness of the tableau method it follows that these syntactic definitions agree with their semantic variants.

Corollary For every theory T and propositions φ, ψ over \mathbb{P} ,

- $T \vdash \varphi$ if and only if $T \models \varphi$,
- $\text{Thm}^{\mathbb{P}}(T) = \theta^{\mathbb{P}}(T)$,
- T is inconsistent if and only if T is unsatisfiable, i.e. it has no model,
- T is complete if and only if T is semantically complete, i.e. it has a single model,
- $T, \varphi \vdash \psi$ if and only if $T \vdash \varphi \rightarrow \psi$ (Deduction theorem).

Remark Deduction theorem can be proved directly by transformations of tableaux.

Theorem on compactness

Theorem A theory T has a model iff every *finite* subset of T has a model.

Proof 1 The implication from left to right is obvious. If T has no model, then it is inconsistent, i.e. \perp is provable by a systematic tableau τ from T . Since τ is finite, \perp is provable from some finite $T' \subseteq T$, i.e. T' has no model. \square

Remark This proof is based on finiteness of proofs, soundness and completeness. We present an alternative proof (applying *König's lemma*).

Proof 2 Let $T = \{\varphi_i \mid i \in \mathbb{N}\}$. Consider a tree S on (certain) finite binary strings σ ordered by being a *prefix*. We put $\sigma \in S$ if and only if there exists an assignment v with prefix σ such that $v \models \varphi_i$ for every $i \leq \text{lth}(\sigma)$.

Observation S has an infinite branch if and only if T has a model.

Since $\{\varphi_i \mid i \leq n\} \subseteq T$ has a model for every $n \in \mathbb{N}$, every level in S is nonempty. Thus S is infinite and moreover binary, hence by König's lemma, S contains an infinite branch. \square

Application of compactness

A graph (V, E) is *k-colorable* if there exists $c: V \rightarrow \{1, \dots, k\}$ such that $c(u) \neq c(v)$ for every edge $\{u, v\} \in E$.

Theorem A countably infinite graph $G = (V, E)$ is *k-colorable* if and only if every *finite subgraph* of G is *k-colorable*.

Proof The implication \Rightarrow is obvious. Assume that every finite subgraph of G is *k-colorable*. Consider $\mathbb{P} = \{p_{u,i} \mid u \in V, 1 \leq i \leq k\}$ and a theory T with axioms

$$\begin{array}{ll} p_{u,1} \vee \dots \vee p_{u,k} & \text{for every } u \in V, \\ \neg(p_{u,i} \wedge p_{u,j}) & \text{for every } u \in V, i < j \leq k, \\ \neg(p_{u,i} \wedge p_{v,i}) & \text{for every } \{u, v\} \in E, i \leq k. \end{array}$$

Then G is *k-colorable* if and only if T has a model. By compactness, it suffices to show that every finite $T' \subseteq T$ has a model. Let G' be the subgraph of G induced by vertices u such that $p_{u,i}$ appears in T' for some i . Since G' is *k-colorable* by the assumption, the theory T' has a model. \square