Propositional and Predicate Logic - VI

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Hilbert's calculus

- basic connectives: $\neg$, $\rightarrow$ (others can be defined from them)
- *logical axioms* (schemes of axioms):

$$
\begin{array}{rc}
(i) & \varphi \rightarrow (\psi \rightarrow \varphi) \\
(ii) & (\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi)) \\
(iii) & (\neg\varphi \rightarrow \neg\psi) \rightarrow (\psi \rightarrow \varphi)
\end{array}
$$

  where φ, ψ, χ are any propositions (of a given language).
- *a rule of inference*:

$$\frac{\varphi,\ \varphi \rightarrow \psi}{\psi} \qquad \text{(modus ponens)}$$

A *proof* (in *Hilbert-style*) of a formula φ from a theory T is a finite sequence $\varphi_0, \ldots, \varphi_n = \varphi$ of formulas such that for every $i \leq n$

- φ_i is a logical axiom or $\varphi_i \in T$ (an axiom of the theory), or
- φ_i can be inferred from the previous formulas applying a rule of inference.

Remark Choice of axioms and inference rules differs in various Hilbert-style proof systems.

Example and soundness

A formula φ is *provable* from T if it has a proof from T, denoted by $T \vdash_H \varphi$.

If $T = \emptyset$, we write $\vdash_H \varphi$. E.g. for $T = \{\neg\varphi\}$ we have $T \vdash_H \varphi \to \psi$ for every ψ.

1)	$\neg\varphi$	an axiom of T
2)	$\neg\varphi \to (\neg\psi \to \neg\varphi)$	a logical axiom (i)
3)	$\neg\psi \to \neg\varphi$	by modus ponens from 1), 2)
4)	$(\neg\psi \to \neg\varphi) \to (\varphi \to \psi)$	a logical axiom (iii)
5)	$\varphi \to \psi$	by modus ponens from 3), 4)

Theorem *For every theory T and formula φ,* $T \vdash_H \varphi \;\Rightarrow\; T \models \varphi$.

Proof

- If φ is an axiom (logical or from T), then $T \models \varphi$ (l. axioms are tautologies),
- if $T \models \varphi$ and $T \models \varphi \to \psi$, then $T \models \psi$, i.e. modus ponens is sound,
- thus every formula in a proof from T is valid in T. $\square$

Remark *The completeness holds as well, i.e.* $T \models \varphi \Rightarrow T \vdash_H \varphi$.

Predicate logic

Deals with statements about objects, their properties and relations.

"She is intelligent and her father knows the rector." $I(x) \wedge K(f(x), r)$

- x is a variable, representing an object,
- r is a constant symbol, representing a particular object,
- f is a function symbol, representing a function,
- I, K are relation (predicate) symbols, representing relations (the property of *"being intelligent"* and the relation *"to know"*).

"Everybody has a father." $(\forall x)(\exists y)(y = f(x))$

- $(\forall x)$ is the universal quantifier (*for every* x),
- $(\exists y)$ is the existential quantifier (*there exists* y),
- $=$ is a (binary) relation symbol, representing the identity relation.

Language

A first-order language consists of

- variables $x, y, z, \ldots, x_0, x_1, \ldots$ (countable many), the set of all variables is denoted by Var,
- function symbols $f, g, h, \ldots$, including constant symbols $c, d, \ldots$, which are nullary function symbols,
- relation (predicate) symbols $P, Q, R, \ldots$, eventually the symbol $=$ (equality) as a special relation symbol,
- quantifiers $(\forall x)$, $(\exists x)$ for every variable $x \in \mathrm{Var}$,
- logical connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$
- parentheses (,)

Every function and relation symbol S has an associated *arity* $\mathrm{ar}(S) \in \mathbb{N}$.

Remark *Compared to propositional logic we have no (explicit) propositional variables, but they can be introduced as nullary relation symbols.*

Signatures

- *Symbols of logic* are variables, quantifiers, connectives and parentheses.
- *Non-logical symbols* are function and relation symbols except the equality symbol. The equality is (usually) considered separately.
- A *signature* is a pair $\langle \mathcal{R}, \mathcal{F} \rangle$ of disjoint sets of relation and function symbols with associated arities, whereas none of them is the equality symbol. A signature lists all non-logical symbols.
- A *language* is determined by a signature $L = \langle \mathcal{R}, \mathcal{F} \rangle$ and by specifying whether it is a language with equality or not. A language must contain at least one relation symbol (non-logical or the equality).

Remark *The meaning of symbols in a language is not assigned, e.g. the symbol* $+$ *does not have to represent the standard addition.*

Examples of languages

We describe a language by a list of all non-logical symbols with eventual clarification of arity and whether they are relation or function symbols.

The following examples of languages are all with equality.

- $L = \langle\ \rangle$ is the language of pure equality,
- $L = \langle c_i \rangle_{i \in \mathbb{N}}$ is the language of countable many constants,
- $L = \langle \leq \rangle$ is the language of orderings,
- $L = \langle E \rangle$ is the language of the graph theory,
- $L = \langle +, -, 0 \rangle$ is the language of the group theory,
- $L = \langle +, -, \cdot, 0, 1 \rangle$ is the language of the field theory,
- $L = \langle -, \wedge, \vee, 0, 1 \rangle$ is the language of Boolean algebras,
- $L = \langle S, +, \cdot, 0, \leq \rangle$ is the language of arithmetic,

where c_i, 0, 1 are constant symbols, S, $-$ are unary function symbols, $+$, $\cdot$, $\wedge$, $\vee$ are binary function symbols, E, $\leq$ are binary relation symbols.

Terms

Are expressions representing values of (composed) functions.

Terms of a language L are defined inductively by

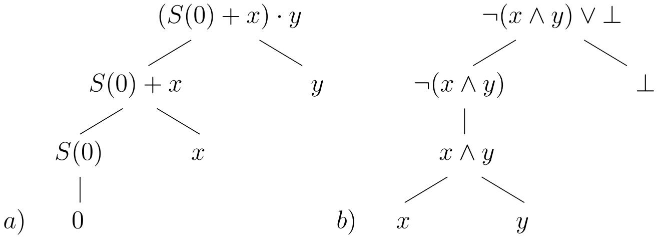
(i) every variable or constant symbol in L is a term,

(ii) if f is a function symbol in L of arity $n > 0$ and $t_1, \ldots, t_n$ are terms, then also the expression $f(t_1, \ldots, t_n)$ is a term,

(iii) every term is formed by a finite number of steps (i), (ii).

- A *ground term* is a term with no variables.
- The set of all terms of a language L is denoted by Term_L.
- A term that is a part of another term t is called a *subterm* of t.
- The structure of terms can be represented by their formation trees.
- For binary function symbols we often use infix notation, e.g. we write $(x + y)$ instead of $+(x, y)$.

Examples of terms



$a)$ The formation tree of the term $(S(0)+x)\cdot y$ of the language of arithmetic.

$b)$ Propositional formulas only with connectives $\neg$, $\wedge$, $\vee$, eventually with constants $\top$, $\perp$ can be viewed as terms of the language of Boolean algebras.

Atomic formulas

Are the simplest formulas.

- An *atomic formula* of a language L is an expression $R(t_1, \dots, t_n)$ where R is an n-ary relation symbol in L and $t_1, \dots, t_n$ are terms of L.
- The set of all atomic formulas of a language L is denoted by AFm_L.
- The structure of an atomic formula can be represented by a formation tree from the formation subtrees of its terms.
- For binary relation symbols we often use infix notation, e.g. $t_1 = t_2$ instead of $=(t_1, t_2)$ or $t_1 \leq t_2$ instead of $\leq(t_1, t_2)$.
- *Examples of atomic formulas*

$$K(f(x), r), \quad x \cdot y \leq (S(0) + x) \cdot y, \quad \neg(x \wedge y) \vee \bot = \bot.$$

Formula

Formulas of a language L are defined inductively by

(i) every atomic formula is a formula,

(ii) if φ, ψ are formulas, then also the following expressions are formulas

$$(\neg\varphi)\ ,(\varphi\wedge\psi)\ ,(\varphi\vee\psi)\ ,(\varphi\rightarrow\psi)\ ,(\varphi\leftrightarrow\psi),$$

(iii) if φ is a formula and x is a variable, then also the expressions $((\forall x)\varphi)$ and $((\exists x)\varphi)$ are formulas.

(iv) every formula is formed by a finite number of steps (i), (ii), (iii).

- The set of all formulas of a language L is denoted by Fm_L.
- A formula that is a part of another formula φ is called a *subformula* of φ.
- The structure of formulas can be represented by their formation trees.

Conventions

- After introducing *priorities* for binary function symbols e.g. $+$, $\cdot$ we are in infix notation allowed to omit parentheses that are around a subterm formed by a symbol of higher priority, e.g. $x \cdot y + z$ instead of $(x \cdot y) + z$.
- After introducing *priorities* for connectives and quantifiers we are allowed to omit parentheses that are around subformulas formed by connectives of higher priority.

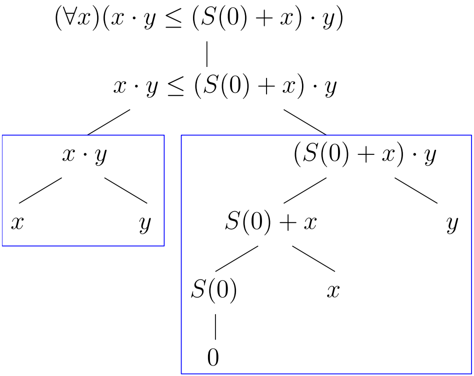
$$(1)\ \neg,\ (\forall x),\ (\exists x) \qquad (2)\ \wedge,\ \vee \qquad (3)\ \rightarrow,\ \leftrightarrow$$

- They can be always omitted around subformulas formed by $\neg$, $(\forall x)$, $(\exists x)$.
- We may also omit parentheses in $(\forall x)$ and $(\exists x)$ for every $x \in \mathrm{Var}$.
- The outer parentheses may be omitted as well.

$$(((\neg((\forall x)R(x))) \wedge ((\exists y)P(y))) \rightarrow (\neg(((\forall x)R(x)) \vee (\neg((\exists y)P(y))))))$$

$$\neg(\forall x)R(x) \wedge (\exists y)P(y) \rightarrow \neg((\forall x)R(x) \vee \neg(\exists y)P(y))$$

An example of a formula



The formation tree of the formula $(\forall x)(x \cdot y \leq (S(0) + x) \cdot y)$.

Occurrences of variables

Let φ be a formula and x be a variable.

- An *occurrence* of x in φ is a leaf labeled by x in the formation tree of φ.
- An occurrence of x in φ is *bound* if it is in some subformula ψ that starts with $(\forall x)$ or $(\exists x)$. An occurrence of x in φ is *free* if it is not bound.
- A variable x is *free* in φ if it has at least one free occurrence in φ. It is *bound* in φ if it has at least one bound occurrence in φ.
- A variable x can be both free and bound in φ. For example in
$$(\forall x)(\exists y)(x \leq y) \vee x \leq z.$$
- We write $\varphi(x_1, \ldots, x_n)$ to denote that $x_1, \ldots, x_n$ are all free variables in the formula φ. *(φ states something about these variables.)*

Remark *We will see that the truth value of a formula (in a given interpretation of symbols) depends only on the assignment of free variables.*

Open and closed formulas

- A formula is *open* if it is without quantifiers. For the set OFm_L of all open formulas in a language L it holds that $\mathrm{AFm}_L \subsetneq \mathrm{OFm}_L \subsetneq \mathrm{Fm}_L$.
- A formula is *closed* (a *sentence*) if it has no free variable; that is, all occurrences of variables are bound.
- A formula can be both open and closed. In this case, all its terms are ground terms.

$x + y \leq 0$	*open*, $\varphi(x, y)$
$(\forall x)(\forall y)(x + y \leq 0)$	*a sentence*,
$(\forall x)(x + y \leq 0)$	*neither open nor a sentence*, $\varphi(y)$
$1 + 0 \leq 0$	*open sentence*

Remark *We will see that in a fixed interpretation of symbols a sentence has a fixed truth value; that is, it does not depend on the assignment of variables.*

Instances

After substituting a term t for a free variable x in a formula φ, we would expect that the new formula (newly) says about t "the same" as φ did about x.

$\varphi(x)$	$(\exists y)(x+y=1)$	*"there is an element $1-x$"*
for $t=1$ we can $\varphi(x/t)$	$(\exists y)(1+y=1)$	*"there is an element $1-1$"*
for $t=y$ we cannot	$(\exists y)(y+y=1)$	*"1 is divisible by 2"*

- A term t is *substitutable* for a variable x in a formula φ if substituting t for all free occurrences of x in φ does not introduce a new bound occurrence of a variable from t.
- Then we denote the obtained formula $\varphi(x/t)$ and we call it an *instance* of the formula φ after a *substitution* of a term t for a variable x.
- t is not substitutable for x in φ if and only if x has a free occurrence in some subformula that starts with $(\forall y)$ or $(\exists y)$ for some variable y in t.
- Ground terms are always substitutable.

Variants

Quantified variables can be (under certain conditions) renamed so that we obtain an equivalent formula.

Let $(Qx)\psi$ be a subformula of φ where Q means $\forall$ or $\exists$ and y is a variable such that the following conditions hold.

1) y is substitutable for x in ψ, and
2) y does not have a free occurrence in ψ.

Then by replacing the subformula $(Qx)\psi$ with $(Qy)\psi(x/y)$ we obtain a *variant* of φ *in subformula* $(Qx)\psi$. After variation of one or more subformulas in φ we obtain a *variant* of φ. *For example,*

$(\exists x)(\forall y)(x \leq y)$	*is a formula* φ,
$(\exists u)(\forall v)(u \leq v)$	*is a variant of* φ,
$(\exists y)(\forall y)(y \leq y)$	*is not a variant of* φ, 1) *does not hold*,
$(\exists x)(\forall x)(x \leq x)$	*is not a variant of* φ, 2) *does not hold*.

Structures

- $\underline{S} = \langle S, \leq \rangle$ is an ordered set where $\leq$ is reflexive, antisymmetric, transitive binary relation on S,
- $G = \langle V, E \rangle$ is an undirected graph without loops where V is the set of *vertices* and E is irreflexive, symmetric binary relation on V *(adjacency)*,
- $\underline{\mathbb{Z}_p} = \langle \mathbb{Z}_p, +, -, 0 \rangle$ is the additive group of integers modulo p,
- $\underline{\mathbb{Q}} = \langle \mathbb{Q}, +, -, \cdot, 0, 1 \rangle$ is the field of rational numbers,
- $\underline{\mathcal{P}(X)} = \langle \mathcal{P}(X), -, \cap, \cup, \emptyset, X \rangle$ is the set algebra over X,
- $\underline{\mathbb{N}} = \langle \mathbb{N}, S, +, \cdot, 0, \leq \rangle$ is the standard model of arithmetic,
- finite automata and other models of computation,
- relational databases, ...

A structure for a language

Let $L = \langle \mathcal{R}, \mathcal{F} \rangle$ be a signature of a language and A be a nonempty set.

- A *realization* (*interpretation*) of a *relation symbol* $R \in \mathcal{R}$ on A is any relation $R^A \subseteq A^{\mathrm{ar}(R)}$. A realization of $=$ on A is the relation Id_A (identity).
- A *realization* (*interpretation*) of a *function symbol* $f \in \mathcal{F}$ on A is any function $f^A \colon A^{\mathrm{ar}(f)} \to A$. Thus a realization of a constant symbol is some element of A.

A *structure* for the language L (*L-structure*) is a triple $\mathcal{A} = \langle A, \mathcal{R}^A, \mathcal{F}^A \rangle$, where

- A is nonempty set, called the *domain* of the structure $\mathcal{A}$,
- $\mathcal{R}^A = \langle R^A \mid R \in \mathcal{R} \rangle$ is a collection of realizations of relation symbols,
- $\mathcal{F}^A = \langle f^A \mid f \in \mathcal{F} \rangle$ is a collection of realizations of function symbols.

A structure for the language L is also called a *model of the language* L. The class of all models of L is denoted by $M(L)$. *Examples for $L = \langle \leq \rangle$ are*

$$\langle \mathbb{N}, \leq \rangle, \ \langle \mathbb{Q}, > \rangle, \ \langle X, E \rangle, \ \langle \mathcal{P}(X), \subseteq \rangle.$$

Value of terms

Let t be a term of $L = \langle \mathcal{R}, \mathcal{F} \rangle$ and $\mathcal{A} = \langle A, \mathcal{R}^A, \mathcal{F}^A \rangle$ be an L-structure.

- A *variable assignment* over the domain A is a function $e\colon \mathrm{Var} \to A$.
- The *value* $t^A[e]$ of the term t in the structure $\mathcal{A}$ with respect to the assignment e is defined by

$$x^A[e] = e(x) \quad \text{for every } x \in \mathrm{Var},$$
$$(f(t_1, \ldots, t_n))^A[e] = f^A(t_1^A[e], \ldots, t_n^A[e]) \quad \text{for every } f \in \mathcal{F}.$$

- In particular, for a constant symbol c we have $c^A[e] = c^A$.
- If t is a ground term, its value in $\mathcal{A}$ is independent on the assignment e.
- The value of t in $\mathcal{A}$ depends only on the assignment of variables in t.

For example, the value of the term $x + 1$ *in the structure* $\mathcal{N} = \langle \mathbb{N}, +, 1 \rangle$ *with respect to the assignment* e *with* $e(x) = 2$ *is* $(x + 1)^N[e] = 3$.

Values of atomic formulas

Let φ be an atomic formula of $L = \langle \mathcal{R}, \mathcal{F} \rangle$ in the form $R(t_1, \ldots, t_n)$, $\mathcal{A} = \langle A, \mathcal{R}^A, \mathcal{F}^A \rangle$ be an L-structure, and e be a variable assignment over A.

- The *value* $H^A_{at}(\varphi)[e]$ of the formula φ in the structure $\mathcal{A}$ with respect to e is

$$H^A_{at}(R(t_1, \ldots, t_n))[e] = \begin{cases} 1 & \text{if } (t_1^A[e], \ldots, t_n^A[e]) \in R^A, \\ 0 & \text{otherwise.} \end{cases}$$

  where $=^A$ is Id_A; that is, $H^A_{at}(t_1 = t_2)[e] = 1$ if $t_1^A[e] = t_2^A[e]$, and $H^A_{at}(t_1 = t_2)[e] = 0$ otherwise.
- If φ is a sentence; that is, all its terms are ground, then its value in $\mathcal{A}$ is independent on the assignment e.
- The value of φ in $\mathcal{A}$ depends only on the assignment of variables in φ.

For example, the value of φ in form $x + 1 \leq 1$ in $\mathcal{N} = \langle \mathbb{N}, +, 1, \leq \rangle$ with respect to the assignment e is $H^{N}_{at}(\varphi)[e] = 1$ if and only if $e(x) = 0$.

Values of formulas

The *value* $H^A(\varphi)[e]$ of the formula φ in the structure $\mathcal{A}$ with respect to e is

$$\begin{aligned}
H^A(\varphi)[e] &= H^A_{at}(\varphi)[e] \ \text{ if } \varphi \text{ is atomic,}\\
H^A(\neg\varphi)[e] &= -_1(H^A(\varphi)[e])\\
H^A(\varphi \wedge \psi)[e] &= \wedge_1(H^A(\varphi)[e], H^A(\psi)[e])\\
H^A(\varphi \vee \psi)[e] &= \vee_1(H^A(\varphi)[e], H^A(\psi)[e])\\
H^A(\varphi \rightarrow \psi)[e] &= \rightarrow_1(H^A(\varphi)[e], H^A(\psi)[e])\\
H^A(\varphi \leftrightarrow \psi)[e] &= \leftrightarrow_1(H^A(\varphi)[e], H^A(\psi)[e])\\
H^A((\forall x)\varphi)[e] &= \min_{a\in A}(H^A(\varphi)[e(x/a)])\\
H^A((\exists x)\varphi)[e] &= \max_{a\in A}(H^A(\varphi)[e(x/a)])
\end{aligned}$$

where $-_1$, $\wedge_1$, $\vee_1$, $\rightarrow_1$, $\leftrightarrow_1$ are the Boolean functions given by the tables and $e(x/a)$ for $a \in A$ denotes the assignment obtained from e by setting $e(x) = a$.

Observation $H^A(\varphi)[e]$ *depends only on the assignment of free variables in* φ.

Satisfiability with respect to assignments

The structure $\mathcal{A}$ satisfies the formula φ with assignment e if $H^A(\varphi)[e] = 1$. Then we write $\mathcal{A} \models \varphi[e]$, and $\mathcal{A} \not\models \varphi[e]$ otherwise. It holds that

$$
\begin{array}{lcl}
\mathcal{A} \models \neg\varphi[e] & \Leftrightarrow & \mathcal{A} \not\models \varphi[e] \\
\mathcal{A} \models (\varphi \wedge \psi)[e] & \Leftrightarrow & \mathcal{A} \models \varphi[e] \ \text{ and } \ \mathcal{A} \models \psi[e] \\
\mathcal{A} \models (\varphi \vee \psi)[e] & \Leftrightarrow & \mathcal{A} \models \varphi[e] \ \text{ or } \ \mathcal{A} \models \psi[e] \\
\mathcal{A} \models (\varphi \rightarrow \psi)[e] & \Leftrightarrow & \mathcal{A} \models \varphi[e] \ \text{ implies } \ \mathcal{A} \models \psi[e] \\
\mathcal{A} \models (\varphi \leftrightarrow \psi)[e] & \Leftrightarrow & \mathcal{A} \models \varphi[e] \ \text{ if and only if } \ \mathcal{A} \models \psi[e] \\
\mathcal{A} \models (\forall x)\varphi[e] & \Leftrightarrow & \mathcal{A} \models \varphi[e(x/a)] \ \text{ for every } a \in A \\
\mathcal{A} \models (\exists x)\varphi[e] & \Leftrightarrow & \mathcal{A} \models \varphi[e(x/a)] \ \text{ for some } a \in A
\end{array}
$$

Observation *Let term t be substitutable for x in φ and ψ be a variant of φ. Then for every structure $\mathcal{A}$ and assignment e*

1) $\mathcal{A} \models \varphi(x/t)[e]$ *if and only if* $\mathcal{A} \models \varphi[e(x/a)]$ *where* $a = t^A[e]$,
2) $\mathcal{A} \models \varphi[e]$ *if and only if* $\mathcal{A} \models \psi[e]$.

Validity in a structure

Let φ be a formula of a language L and $\mathcal{A}$ be an L-structure.

- φ is *valid* (*true*) *in the structure* $\mathcal{A}$, denoted by $\mathcal{A} \models \varphi$, if $\mathcal{A} \models \varphi[e]$ for every $e\colon \mathrm{Var} \to A$. We say that $\mathcal{A}$ *satisfies* φ. Otherwise, we write $\mathcal{A} \not\models \varphi$.
- φ is *contradictory in* $\mathcal{A}$ if $\mathcal{A} \models \neg\varphi$; that is, $\mathcal{A} \not\models \varphi[e]$ for every $e\colon \mathrm{Var} \to A$.
- For every formulas φ, ψ, variable x, and structure $\mathcal{A}$

$$\begin{array}{llcl}
(1) & \mathcal{A} \models \varphi & \Rightarrow & \mathcal{A} \not\models \neg\varphi \\
(2) & \mathcal{A} \models \varphi \wedge \psi & \Leftrightarrow & \mathcal{A} \models \varphi \text{ and } \mathcal{A} \models \psi \\
(3) & \mathcal{A} \models \varphi \vee \psi & \Leftarrow & \mathcal{A} \models \varphi \text{ or } \mathcal{A} \models \psi \\
(4) & \mathcal{A} \models \varphi & \Leftrightarrow & \mathcal{A} \models (\forall x)\varphi
\end{array}$$

- If φ is a sentence, it is valid or contradictory in $\mathcal{A}$, and thus (1) holds also in $\Leftarrow$. If moreover ψ is a sentence, also (3) holds in $\Rightarrow$.
- By (4), $\mathcal{A} \models \varphi$ if and only if $\mathcal{A} \models \psi$ where ψ is a *universal closure* of φ, i.e. a formula $(\forall x_1) \cdots (\forall x_n)\varphi$ where $x_1, \ldots, x_n$ are all free variables in φ.

Validity in a theory

- A *theory* of language L is any set T of formulas of L (so called *axioms*).
- A *model of a theory* T is an L-structure $\mathcal{A}$ such that $\mathcal{A} \models \varphi$ for every $\varphi \in T$. Then we write $\mathcal{A} \models T$ and we say that $\mathcal{A}$ *satisfies* T.
- The *class of models* of a theory T is $M(T) = \{\mathcal{A} \in M(L) \mid \mathcal{A} \models T\}$.
- A formula φ is *valid in T* (*true in T*), denoted by $T \models \varphi$, if $\mathcal{A} \models \varphi$ for every model $\mathcal{A}$ of T. Otherwise, we write $T \not\models \varphi$.
- φ is *contradictory in T* if $T \models \neg\varphi$, i.e. φ is contradictory in all models of T.
- φ is *independent in T* if it is neither valid nor contradictory in T.
- If $T = \emptyset$, we have $M(T) = M(L)$ and we omit T, eventually we say *"in logic"*. Then $\models \varphi$ means that φ is (*universally*) *valid* (a *tautology*).
- A *consequence* of T is the set $\theta^L(T)$ of all sentences of L valid in T, i.e.

$$\theta^L(T) = \{\varphi \in \mathrm{Fm}_L \mid T \models \varphi \text{ and } \varphi \text{ is a sentence}\}.$$

Example of a theory

A *theory of orderings* T in language $L = \langle \leq \rangle$ with equality has axioms

$$x \leq x \qquad \text{(reflexivity)}$$

$$x \leq y \;\wedge\; y \leq x \;\rightarrow\; x = y \qquad \text{(antisymmetry)}$$

$$x \leq y \;\wedge\; y \leq z \;\rightarrow\; x \leq z \qquad \text{(transitivity)}$$

Models of T are L-structures $\langle S, \leq_S \rangle$, so called ordered sets, that satisfy the axioms of T, for example $\mathcal{A} = \langle \mathbb{N}, \leq \rangle$ or $\mathcal{B} = \langle \mathcal{P}(X), \subseteq \rangle$ for $X = \{0, 1, 2\}$.

- *A formula* $\varphi\colon x \leq y \vee y \leq x$ *is valid in* $\mathcal{A}$ *but not in* $\mathcal{B}$ *since* $\mathcal{B} \not\models \varphi[e]$ *for the assignment* $e(x) = \{0\}, e(y) = \{1\}$*, thus* φ *is independent in* T*.*
- *A sentence* $\psi\colon (\exists x)(\forall y)(y \leq x)$ *is valid in* $\mathcal{B}$ *and contradictory in* $\mathcal{A}$*, hence it is independent in* T *as well. We write* $\mathcal{B} \models \psi$, $\mathcal{A} \models \neg\psi$.
- *A formula* $\chi\colon (x \leq y \wedge y \leq z \wedge z \leq x) \rightarrow (x = y \wedge y = z)$ *is valid in* T*, denoted by* $T \models \chi$*, the same holds for its* *universal closure.*