Propositional and Predicate Logic - VII

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WS 2024/2025

Value of terms

Let t be a term of $L = \langle \mathcal{R}, \mathcal{F} \rangle$ and $\mathcal{A} = \langle A, \mathcal{R}^A, \mathcal{F}^A \rangle$ be an L-structure.

- A *variable assignment* over the domain A is a function $e\colon \mathrm{Var} \to A$.
- The *value* $t^A[e]$ of the term t in the structure $\mathcal{A}$ with respect to the assignment e is defined by

$$x^A[e] = e(x) \quad \text{for every } x \in \mathrm{Var},$$
$$(f(t_1, \dots, t_n))^A[e] = f^A(t_1^A[e], \dots, t_n^A[e]) \quad \text{for every } f \in \mathcal{F}.$$

- In particular, for a constant symbol c we have $c^A[e] = c^A$.
- If t is a ground term, its value in $\mathcal{A}$ is independent on the assignment e.
- The value of t in $\mathcal{A}$ depends only on the assignment of variables in t.

For example, the value of the term $x + 1$ in the structure $\mathcal{N} = \langle \mathbb{N}, +, 1 \rangle$ with respect to the assignment e with $e(x) = 2$ is $(x + 1)^N[e] = 3$.

Values of atomic formulas

Let φ be an atomic formula of $L = \langle \mathcal{R}, \mathcal{F} \rangle$ in the form $R(t_1, \ldots, t_n)$, $\mathcal{A} = \langle A, \mathcal{R}^A, \mathcal{F}^A \rangle$ be an L-structure, and e be a variable assignment over A.

- The *value* $H^A_{at}(\varphi)[e]$ of the formula φ in the structure $\mathcal{A}$ with respect to e is

$$H^A_{at}(R(t_1, \ldots, t_n))[e] = \begin{cases} 1 & \text{if } (t_1^A[e], \ldots, t_n^A[e]) \in R^A, \\ 0 & \text{otherwise.} \end{cases}$$

  where $=^A$ is Id_A; that is, $H^A_{at}(t_1 = t_2)[e] = 1$ if $t_1^A[e] = t_2^A[e]$, and $H^A_{at}(t_1 = t_2)[e] = 0$ otherwise.
- If φ is a sentence; that is, all its terms are ground, then its value in $\mathcal{A}$ is independent on the assignment e.
- The value of φ in $\mathcal{A}$ depends only on the assignment of variables in φ.

For example, the value of φ in form $x + 1 \leq 1$ in $\mathcal{N} = \langle \mathbb{N}, +, 1, \leq \rangle$ with respect to the assignment e is $H^N_{at}(\varphi)[e] = 1$ if and only if $e(x) = 0$.

Values of formulas

The *value* $H^A(\varphi)[e]$ of the formula φ in the structure $\mathcal{A}$ with respect to e is

$$\begin{aligned}
H^A(\varphi)[e] &= H^A_{at}(\varphi)[e] \ \text{ if } \varphi \text{ is atomic,}\\
H^A(\neg\varphi)[e] &= -_1(H^A(\varphi)[e])\\
H^A(\varphi \wedge \psi)[e] &= \wedge_1(H^A(\varphi)[e], H^A(\psi)[e])\\
H^A(\varphi \vee \psi)[e] &= \vee_1(H^A(\varphi)[e], H^A(\psi)[e])\\
H^A(\varphi \to \psi)[e] &= \to_1(H^A(\varphi)[e], H^A(\psi)[e])\\
H^A(\varphi \leftrightarrow \psi)[e] &= \leftrightarrow_1(H^A(\varphi)[e], H^A(\psi)[e])\\
H^A((\forall x)\varphi)[e] &= \min_{a\in A}(H^A(\varphi)[e(x/a)])\\
H^A((\exists x)\varphi)[e] &= \max_{a\in A}(H^A(\varphi)[e(x/a)])
\end{aligned}$$

where $-_1$, $\wedge_1$, $\vee_1$, $\to_1$, $\leftrightarrow_1$ are the Boolean functions given by the tables and $e(x/a)$ for $a \in A$ denotes the assignment obtained from e by setting $e(x) = a$.

Observation $H^A(\varphi)[e]$ *depends only on the assignment of free variables in* φ.

Satisfiability with respect to assignments

The structure $\mathcal{A}$ satisfies the formula φ with assignment e if $H^A(\varphi)[e] = 1$. Then we write $\mathcal{A} \models \varphi[e]$, and $\mathcal{A} \not\models \varphi[e]$ otherwise. It holds that

$$
\begin{array}{lcl}
\mathcal{A} \models \neg\varphi[e] & \Leftrightarrow & \mathcal{A} \not\models \varphi[e] \\
\mathcal{A} \models (\varphi \wedge \psi)[e] & \Leftrightarrow & \mathcal{A} \models \varphi[e] \text{ and } \mathcal{A} \models \psi[e] \\
\mathcal{A} \models (\varphi \vee \psi)[e] & \Leftrightarrow & \mathcal{A} \models \varphi[e] \text{ or } \mathcal{A} \models \psi[e] \\
\mathcal{A} \models (\varphi \rightarrow \psi)[e] & \Leftrightarrow & \mathcal{A} \models \varphi[e] \text{ implies } \mathcal{A} \models \psi[e] \\
\mathcal{A} \models (\varphi \leftrightarrow \psi)[e] & \Leftrightarrow & \mathcal{A} \models \varphi[e] \text{ if and only if } \mathcal{A} \models \psi[e] \\
\mathcal{A} \models (\forall x)\varphi[e] & \Leftrightarrow & \mathcal{A} \models \varphi[e(x/a)] \text{ for every } a \in A \\
\mathcal{A} \models (\exists x)\varphi[e] & \Leftrightarrow & \mathcal{A} \models \varphi[e(x/a)] \text{ for some } a \in A
\end{array}
$$

Observation *Let term t be substitutable for x in φ and ψ be a variant of φ. Then for every structure $\mathcal{A}$ and assignment e*

1) $\mathcal{A} \models \varphi(x/t)[e]$ *if and only if* $\mathcal{A} \models \varphi[e(x/a)]$ *where* $a = t^A[e]$,
2) $\mathcal{A} \models \varphi[e]$ *if and only if* $\mathcal{A} \models \psi[e]$.

Validity in a structure

Let φ be a formula of a language L and $\mathcal{A}$ be an L-structure.

- φ is *valid* (*true*) *in the structure* $\mathcal{A}$, denoted by $\mathcal{A} \models \varphi$, if $\mathcal{A} \models \varphi[e]$ for every $e\colon \mathrm{Var} \to A$. We say that $\mathcal{A}$ *satisfies* φ. Otherwise, we write $\mathcal{A} \not\models \varphi$.
- φ is *contradictory in* $\mathcal{A}$ if $\mathcal{A} \models \neg\varphi$; that is, $\mathcal{A} \not\models \varphi[e]$ for every $e\colon \mathrm{Var} \to A$.
- For every formulas φ, ψ, variable x, and structure $\mathcal{A}$

$$
\begin{array}{llcl}
(1) & \mathcal{A} \models \varphi & \Rightarrow & \mathcal{A} \not\models \neg\varphi \\
(2) & \mathcal{A} \models \varphi \wedge \psi & \Leftrightarrow & \mathcal{A} \models \varphi \text{ and } \mathcal{A} \models \psi \\
(3) & \mathcal{A} \models \varphi \vee \psi & \Leftarrow & \mathcal{A} \models \varphi \text{ or } \mathcal{A} \models \psi \\
(4) & \mathcal{A} \models \varphi & \Leftrightarrow & \mathcal{A} \models (\forall x)\varphi
\end{array}
$$

- If φ is a sentence, it is valid or contradictory in $\mathcal{A}$, and thus (1) holds also in $\Leftarrow$. If moreover ψ is a sentence, also (3) holds in $\Rightarrow$.
- By (4), $\mathcal{A} \models \varphi$ if and only if $\mathcal{A} \models \psi$ where ψ is a *universal closure* of φ, i.e. a formula $(\forall x_1)\cdots(\forall x_n)\varphi$ where $x_1, \ldots, x_n$ are all free variables in φ.

Validity in a theory

- A *theory* of a language L is any set T of formulas of L (so called *axioms*).
- A *model of a theory* T is an L-structure $\mathcal{A}$ such that $\mathcal{A} \models \varphi$ for every $\varphi \in T$. Then we write $\mathcal{A} \models T$ and we say that $\mathcal{A}$ *satisfies* T.
- The *class of models* of a theory T is $M(T) = \{\mathcal{A} \in M(L) \mid \mathcal{A} \models T\}$.
- A formula φ is *valid in T* (*true in T*), denoted by $T \models \varphi$, if $\mathcal{A} \models \varphi$ for every model $\mathcal{A}$ of T. Otherwise, we write $T \not\models \varphi$.
- φ is *contradictory in T* if $T \models \neg\varphi$, i.e. φ is contradictory in all models of T.
- φ is *independent in T* if it is neither valid nor contradictory in T.
- If $T = \emptyset$, we have $M(T) = M(L)$ and we omit T, eventually we say *"in logic"*. Then $\models \varphi$ means that φ is (*logically*) *valid* (a *tautology*).
- A *consequence* of T is the set $\theta^L(T)$ of all sentences of L valid in T, i.e.

$$\theta^L(T) = \{\varphi \in \mathrm{Fm}_L \mid T \models \varphi \text{ and } \varphi \text{ is a sentence}\}.$$

Example of a theory

The *theory of orderings* T of the language $L = \langle \leq \rangle$ with equality has axioms

$$x \leq x \qquad \text{(reflexivity)}$$
$$x \leq y \;\wedge\; y \leq x \;\rightarrow\; x = y \qquad \text{(antisymmetry)}$$
$$x \leq y \;\wedge\; y \leq z \;\rightarrow\; x \leq z \qquad \text{(transitivity)}$$

Models of T are L-structures $\langle S, \leq_S \rangle$, so called ordered sets, that satisfy the axioms of T, for example $\mathcal{A} = \langle \mathbb{N}, \leq \rangle$ or $\mathcal{B} = \langle \mathcal{P}(X), \subseteq \rangle$ for $X = \{0, 1, 2\}$.

- *The formula* $\varphi\colon\ x \leq y \vee y \leq x$ *is valid in* $\mathcal{A}$ *but not in* $\mathcal{B}$ *since* $\mathcal{B} \not\models \varphi[e]$ *for the assignment* $e(x) = \{0\}, e(y) = \{1\}$, *thus* φ *is independent in* T.
- *The sentence* $\psi\colon\ (\exists x)(\forall y)(y \leq x)$ *is valid in* $\mathcal{B}$ *and contradictory in* $\mathcal{A}$, *hence it is independent in* T *as well. We write* $\mathcal{B} \models \psi$, $\mathcal{A} \models \neg\psi$.
- *The formula* $\chi\colon\ (x \leq y \wedge y \leq z \wedge z \leq x) \rightarrow (x = y \wedge y = z)$ *is valid in* T, *denoted by* $T \models \chi$, *the same holds for its universal closure.*

Unsatisfiability and validity

The problem of validity in a theory can be transformed to the problem of satisfiability of (another) theory.

Proposition *For every theory T and sentence φ (of the same language)*

$$T, \neg\varphi \text{ is unsatisfiable} \quad \Leftrightarrow \quad T \models \varphi.$$

Proof By definitions, it is equivalent that

(1) $T, \neg\varphi$ is unsatisfiable (i.e. it has no model),

(2) $\neg\varphi$ is not valid in any model of T,

(3) φ is valid in every model of T,

(4) $T \models \varphi$. $\square$

Remark *The assumption that φ is a sentence is necessary for $(2) \Rightarrow (3)$.*

For example, the theory $\{P(c), \neg P(x)\}$ is unsatisfiable, but $P(c) \not\models P(x)$, where P is a unary relation symbol and c is a constant symbol.

Basic algebraic theories

- theory of *groups* in the language $L = \langle +, -, 0 \rangle$ with equality has axioms

$$x + (y + z) = (x + y) + z \qquad \text{(associativity of } +\text{)}$$
$$0 + x = x = x + 0 \qquad \text{(0 is neutral to } +\text{)}$$
$$x + (-x) = 0 = (-x) + x \qquad (-x \text{ is inverse of } x)$$

- theory of *Abelian groups* has moreover ax. $x + y = y + x$ (commutativity)
- theory of *rings* in $L = \langle +, -, \cdot, 0, 1 \rangle$ with equality has moreover axioms

$$1 \cdot x = x = x \cdot 1 \qquad \text{(1 is neutral to } \cdot\text{)}$$
$$x \cdot (y \cdot z) = (x \cdot y) \cdot z \qquad \text{(associativity of } \cdot\text{)}$$
$$x \cdot (y + z) = x \cdot y + x \cdot z,\ (x + y) \cdot z = x \cdot z + y \cdot z \qquad \text{(distributivity)}$$

- theory of *commutative rings* has moreover ax. $x \cdot y = y \cdot x$ (commutativity)
- theory of *fields* in the same language has additional axioms

$$x \neq 0 \rightarrow (\exists y)(x \cdot y = 1) \qquad \text{(existence of inverses to } \cdot\text{)}$$
$$0 \neq 1 \qquad \text{(nontriviality)}$$

Properties of theories

A theory T of a language L is *(semantically)*

- *inconsistent* if $T \models \perp$, otherwise T is *consistent* (*satisfiable*),
- *complete* if it is consistent and every sentence of L is valid in T or contradictory in T,
- an *extension* of a theory T' of language L' if $L' \subseteq L$ and $\theta^{L'}(T') \subseteq \theta^{L}(T)$, we say that an extension T of a theory T' is *simple* if $L = L'$; and *conservative* if $\theta^{L'}(T') = \theta^{L}(T) \cap \mathrm{Fm}_{L'}$,
- *equivalent* with a theory T' if T is an extension of T' and vice-versa,

Structures $\mathcal{A}$, $\mathcal{B}$ for a language L are *elementarily equivalent*, denoted by $\mathcal{A} \equiv \mathcal{B}$, if they satisfy the same sentences of L.

Observation *Let T and T' be theories of a language L. T is (semantically)*

(1) *consistent if and only if it has a model,*

(2) *complete iff it has a single model, up to elementarily equivalence,*

(3) *an extension of T' if and only if $M(T) \subseteq M(T')$,*

(4) *equivalent with T' if and only if $M(T) = M(T')$.*

Substructures

Let $\mathcal{A} = \langle A, \mathcal{R}^A, \mathcal{F}^A \rangle$ and $\mathcal{B} = \langle B, \mathcal{R}^B, \mathcal{F}^B \rangle$ be structures for $L = \langle \mathcal{R}, \mathcal{F} \rangle$.

We say that $\mathcal{B}$ is an (induced) *substructure* of $\mathcal{A}$, denoted by $\mathcal{B} \subseteq \mathcal{A}$, if

(i) $B \subseteq A$,

(ii) $R^B = R^A \cap B^{\mathrm{ar}(R)}$ for every $R \in \mathcal{R}$,

(iii) $f^B = f^A \cap (B^{\mathrm{ar}(f)} \times B)$; that is, $f^B = f^A \restriction B^{\mathrm{ar}(f)}$, for every $f \in \mathcal{F}$.

A set $C \subseteq A$ is a domain of some substructure of $\mathcal{A}$ if and only if C is closed under all functions of $\mathcal{A}$. Then the respective substructure, denoted by $\mathcal{A} \restriction C$, is said to be the *restriction* of the structure $\mathcal{A}$ to C.

- A set $C \subseteq A$ is *closed* under a function $f\colon A^n \to A$ if $f(x_1, \ldots, x_n) \in C$ for every $x_1, \ldots, x_n \in C$.

Example: $\underline{\mathbb{Z}} = \langle \mathbb{Z}, +, \cdot, 0 \rangle$ *is a substructure of* $\underline{\mathbb{Q}} = \langle \mathbb{Q}, +, \cdot, 0 \rangle$ *and* $\underline{\mathbb{Z}} = \underline{\mathbb{Q}} \restriction \mathbb{Z}$. *Furthermore,* $\underline{\mathbb{N}} = \langle \mathbb{N}, +, \cdot, 0 \rangle$ *is their substructure and* $\underline{\mathbb{N}} = \underline{\mathbb{Q}} \restriction \mathbb{N} = \underline{\mathbb{Z}} \restriction \mathbb{N}$.

Validity in a substructure

Let $\mathcal{B}$ be a substructure of a structure $\mathcal{A}$ for a (fixed) language L.

Proposition *For every open formula φ and assignment $e\colon \mathrm{Var} \to B$,*

$$\mathcal{A} \models \varphi[e] \quad \text{if and only if} \quad \mathcal{B} \models \varphi[e].$$

Proof For atomic φ it follows from the definition of the truth value with respect to an assignment. Otherwise by induction on the structure of the formula. □

Corollary *For every open formula φ and structure $\mathcal{A}$,*

$$\mathcal{A} \models \varphi \quad \text{if and only if} \quad \mathcal{B} \models \varphi \text{ for every substructure } \mathcal{B} \subseteq \mathcal{A}.$$

- A theory T is *open* if all axioms of T are open.

Corollary *Every substructure of a model of an open theory T is a model of T.*

For example, every substructure of a graph, i.e. a model of theory of graphs, is a graph, called a subgraph. Similarly subgroups, Boolean subalgebras, etc.

Generated substructure, expansion, reduct

Let $\mathcal{A} = \langle A, \mathcal{R}^A, \mathcal{F}^A \rangle$ be a structure and $X \subseteq A$. Let B be the smallest subset of A containing X that is closed under all functions of the structure $\mathcal{A}$ (including constants). Then the structure $\mathcal{A} \upharpoonright B$ is denoted by $\mathcal{A}\langle X \rangle$ and is called the substructure of $\mathcal{A}$ *generated* by the set X.

Example: for $\underline{\mathbb{Q}} = \langle \mathbb{Q}, +, \cdot, 0 \rangle$, $\underline{\mathbb{Z}} = \langle \mathbb{Z}, +, \cdot, 0 \rangle$, $\underline{\mathbb{N}} = \langle \mathbb{N}, +, \cdot, 0 \rangle$ *it is* $\underline{\mathbb{Q}}\langle \{1\} \rangle = \underline{\mathbb{N}}$, $\underline{\mathbb{Q}}\langle \{-1\} \rangle = \underline{\mathbb{Z}}$, *and* $\underline{\mathbb{Q}}\langle \{2\} \rangle$ *is the substructure on all even natural numbers.*

Let $\mathcal{A}$ be a structure for a language L and $L' \subseteq L$. By omitting realizations of symbols that are not in L' we obtain from $\mathcal{A}$ a structure $\mathcal{A}'$ called the *reduct* of $\mathcal{A}$ to the language L'. Conversely, $\mathcal{A}$ is an *expansion* of $\mathcal{A}'$ into L.

For example, $\langle \mathbb{N}, + \rangle$ *is a reduct of* $\langle \mathbb{N}, +, \cdot, 0 \rangle$. *On the other hand, the structure* $\langle \mathbb{N}, +, c_i \rangle_{i \in \mathbb{N}}$ *with* $c_i = i$ *for every* $i \in \mathbb{N}$ *is the expansion of* $\langle \mathbb{N}, + \rangle$ *by names of elements from* $\mathbb{N}$.

Theorem on constants

Theorem *Let φ be a formula in a language L with free variables $x_1, \ldots, x_n$ and let T be a theory in L. Let L' be the extension of L with new constant symbols $c_1, \ldots, c_n$ and let T' denote the theory T in L'. Then*

$$T \models \varphi \quad \textit{if and only if} \quad T' \models \varphi(x_1/c_1, \ldots, x_n/c_n).$$

Proof ($\Rightarrow$) If $\mathcal{A}'$ is a model of T', let $\mathcal{A}$ be the reduct of $\mathcal{A}'$ to L. Since $\mathcal{A} \models \varphi[e]$ for every assignment e, we have in particular

$$\mathcal{A} \models \varphi[e(x_1/c_1^{A'}, \ldots, x_n/c_n^{A'})], \quad \text{i.e.} \;\; \mathcal{A}' \models \varphi(x_1/c_1, \ldots, x_n/c_n).$$

($\Leftarrow$) If $\mathcal{A}$ is a model of T and e an assignment, let $\mathcal{A}'$ be the expansion of A into L' by setting $c_i^{A'} = e(x_i)$ for every i. Since $\mathcal{A}' \models \varphi(x_1/c_1, \ldots, x_n/c_n)[e']$ for every assignment e', we have

$$\mathcal{A}' \models \varphi[e(x_1/c_1^{A'}, \ldots, x_n/c_n^{A'})], \quad \text{i.e.} \;\; \mathcal{A} \models \varphi[e]. \quad \square$$

Extensions of theories

Proposition *Let T be a theory of L and T' be a theory of L' where $L \subseteq L'$.*

(i) *T' is an extension of T if and only if the reduct $\mathcal{A}$ of every model $\mathcal{A}'$ of T' to the language L is a model of T,*

(ii) *T' is a conservative extension of T if T' is an extension of T and every model $\mathcal{A}$ of T can be expanded to the language L' on a model $\mathcal{A}'$ of T'.*

Proof

$(i)a)$ If T' is an extension of T and φ is any axiom of T, then $T' \models \varphi$. Thus $\mathcal{A}' \models \varphi$ and also $\mathcal{A} \models \varphi$, which implies that $\mathcal{A}$ is a model of T.

$(i)b)$ If $\mathcal{A}$ is a model of T and $T \models \varphi$ where φ is of L, then $\mathcal{A} \models \varphi$ and also $\mathcal{A}' \models \varphi$. This implies that $T' \models \varphi$ and thus T' is an extension of T.

(ii) If $T' \models \varphi$ where φ is of L and $\mathcal{A}$ is a model of T, then in its expansion $\mathcal{A}'$ that models T' we have $\mathcal{A}' \models \varphi$. Thus also $\mathcal{A} \models \varphi$, and hence $T \models \varphi$. Therefore T' is conservative. $\square$