

Propositional and Predicate Logic - VIII

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Extensions of theories

Proposition Let T be a theory of L and T' be a theory of L' where $L \subseteq L'$.

- (i) T' is an extension of T if and only if the **reduct** \mathcal{A} of every model \mathcal{A}' of T' to the language L is a model of T ,
- (ii) T' is a **conservative** extension of T if T' is an extension of T and every model \mathcal{A} of T can be **expanded** to the language L' on a model \mathcal{A}' of T' .

Proof

- (i)a) If T' is an extension of T and φ is any axiom of T , then $T' \models \varphi$. Thus $\mathcal{A}' \models \varphi$ and also $\mathcal{A} \models \varphi$, which implies that \mathcal{A} is a model of T .
- (i)b) If \mathcal{A} is a model of T and $T \models \varphi$ where φ is of L , then $\mathcal{A} \models \varphi$ and also $\mathcal{A}' \models \varphi$. This implies that $T' \models \varphi$ and thus T' is an extension of T .
- (ii) If $T' \models \varphi$ where φ is of L and \mathcal{A} is a model of T , then in its expansion \mathcal{A}' that models T' we have $\mathcal{A}' \models \varphi$. Thus also $\mathcal{A} \models \varphi$, and hence $T \models \varphi$. Therefore T' is conservative. \square

Extensions by definition of a relation symbol

Let T be a theory of L , $\psi(x_1, \dots, x_n)$ be a formula of L in free variables x_1, \dots, x_n and L' denote the language L with a new n -ary relation symbol R .

The *extension* of T *by definition of R* with the formula ψ is the theory T' of L' obtained from T by adding the axiom

$$R(x_1, \dots, x_n) \leftrightarrow \psi(x_1, \dots, x_n)$$

Observation Every model of T can be *uniquely* expanded to a model of T' .

Corollary T' is a *conservative* extension of T .

Proposition For every formula φ' of L' there is φ of L s.t. $T' \models \varphi' \leftrightarrow \varphi$.

Proof Replace each subformula $R(t_1, \dots, t_n)$ in φ' with $\psi'(x_1/t_1, \dots, x_n/t_n)$, where ψ' is a suitable variant of ψ allowing all substitutions. \square

For example, the symbol \leq can be defined in arithmetics by the axiom

$$x \leq y \leftrightarrow (\exists z)(x + z = y)$$

Extensions by definition of a function symbol

Let T be a theory of a language L and $\psi(x_1, \dots, x_n, y)$ be a formula of L in free variables x_1, \dots, x_n, y such that

$$T \models (\exists y)\psi(x_1, \dots, x_n, y) \quad \text{(existence)}$$

$$T \models \psi(x_1, \dots, x_n, y) \wedge \psi(x_1, \dots, x_n, z) \rightarrow y = z \quad \text{(uniqueness)}$$

Let L' denote the language L with a new n -ary function symbol f .

The *extension* of T *by definition of f* with the formula ψ is the theory T' of L' obtained from T by adding the axiom

$$f(x_1, \dots, x_n) = y \leftrightarrow \psi(x_1, \dots, x_n, y)$$

Remark In particular, if ψ is $t(x_1, \dots, x_n) = y$ where t is a term and x_1, \dots, x_n are the variables in t , both the conditions of existence and uniqueness hold.

For example binary $-$ can be defined using $+$ and unary $-$ by the axiom

$$x - y = z \leftrightarrow x + (-y) = z$$

Extensions by definition of a function symbol (cont.)

Observation Every model of T can be *uniquely* expanded to a model of T' .

Corollary T' is a *conservative* extension of T .

Proposition For every formula φ' of L' there is φ of L s.t. $T' \models \varphi' \leftrightarrow \varphi$.

Proof It suffices to consider φ' with a single occurrence of f . If φ' has more, we may proceed inductively. Let φ^* denote the formula obtained from φ' by replacing the term $f(t_1, \dots, t_n)$ with a **new** variable z . Let φ be the formula

$$(\exists z)(\varphi^* \wedge \psi'(x_1/t_1, \dots, x_n/t_n, y/z)),$$

where ψ' is a suitable variant of ψ allowing all substitutions.

Let \mathcal{A} be a model of T' , e be an assignment, and $a = f^{\mathcal{A}}(t_1, \dots, t_n)[e]$. By the two conditions, $\mathcal{A} \models \psi'(x_1/t_1, \dots, x_n/t_n, y/z)[e]$ if and only if $e(z) = a$. Thus

$$\mathcal{A} \models \varphi[e] \Leftrightarrow \mathcal{A} \models \varphi^*[e(z/a)] \Leftrightarrow \mathcal{A} \models \varphi'[e]$$

for every assignment e , i.e. $\mathcal{A} \models \varphi' \leftrightarrow \varphi$ and so $T' \models \varphi' \leftrightarrow \varphi$. \square

Extensions by definitions

A theory T' of L' is called an *extension* of a theory T of L *by definitions* if it is obtained from T by successive definitions of relation and function symbols.

Corollary *Let T' be an extension of a theory T by definitions. Then*

- every model of T can be *uniquely* expanded to a model of T' ,
- T' is a *conservative* extension of T ,
- for every formula φ' of L' there is a formula φ of L such that $T' \models \varphi' \leftrightarrow \varphi$.

For example, in $T = \{(\exists y)(x + y = 0), (x + y = 0) \wedge (x + z = 0) \rightarrow y = z\}$ of $L = \langle +, 0, \leq \rangle$ with equality we can define $<$ and unary $-$ by the axioms

$$\begin{aligned} -x = y &\leftrightarrow x + y = 0 \\ x < y &\leftrightarrow x \leq y \wedge \neg(x = y) \end{aligned}$$

Then the formula $-x < y$ is equivalent in this extension to a formula

$$(\exists z)((z \leq y \wedge \neg(z = y)) \wedge x + z = 0).$$

Definable sets

We are interested in which sets can be defined within a given structure.

- A set defined by a formula $\varphi(x_1, \dots, x_n)$ in structure \mathcal{A} is the set

$$\varphi^{\mathcal{A}}(x_1, \dots, x_n) = \{(a_1, \dots, a_n) \in A^n \mid \mathcal{A} \models \varphi[e(x_1/a_1, \dots, x_n/a_n)]\}.$$

Shortly, $\varphi^{\mathcal{A}}(\bar{x}) = \{\bar{a} \in A^{|\bar{x}|} \mid \mathcal{A} \models \varphi[e(\bar{x}/\bar{a})]\}$, where $|\bar{x}| = n$.

- A set defined by a formula $\varphi(\bar{x}, \bar{y})$ with parameters $\bar{b} \in A^{|\bar{y}|}$ in \mathcal{A} is

$$\varphi^{\mathcal{A}, \bar{b}}(\bar{x}, \bar{y}) = \{\bar{a} \in A^{|\bar{x}|} \mid \mathcal{A} \models \varphi[e(\bar{x}/\bar{a}, \bar{y}/\bar{b})]\}.$$

Example: $E(x, y)^{\mathcal{G}, b}$ is the set of neighbors of a vertex b in a graph \mathcal{G} .

- For a structure \mathcal{A} , a set $B \subseteq A$, and $n \in \mathbb{N}$ let $\text{Df}^n(\mathcal{A}, B)$ denote the class of definable sets $D \subseteq A^n$ in the structure \mathcal{A} with parameters from B .

Observation $\text{Df}^n(\mathcal{A}, B)$ is closed under complements, union, intersection and it contains \emptyset, A^n . Thus it forms a subalgebra of the set algebra $\underline{\mathcal{P}}(A^n)$.

Example - database queries

<i>Movie</i>	<i>name</i>	<i>director</i>	<i>actor</i>	<i>Program</i>	<i>cinema</i>	<i>name</i>	<i>time</i>
	Lidé z Maringotek	M. Frič	J. Tříška		Světozor	Po strništi bos	13:15
	Po strništi bos	J. Svěrák	Z. Svěrák		Mat	Po strništi bos	16:15
	Po strništi bos	J. Svěrák	J. Tříška		Mat	Lidé z Maringotek	18:30

Where and when can I see a movie with J. Tříška?

select *Program.cinema*, *Program.time* **from** *Movie*, *Program*
where *Movie.name* = *Program.name* **and** *actor* = 'J. Tříška';

Equivalently, it is the set $\varphi^{\mathcal{D}}(x, y)$ defined by the formula $\varphi(x, y)$

$$(\exists n)(\exists d)(P(x, n, y) \wedge M(n, d, \text{'J. Tříška'}))$$

in the structure $\mathcal{D} = \langle D, \textit{Movie}, \textit{Program}, c^{\mathcal{D}} \rangle_{c \in D}$ of $L = \langle M, P, c \rangle_{c \in D}$, where $D = \{\text{'Po strništi bos'}, \text{'J. Tříška'}, \text{'Mat'}, \text{'13:15'}, \dots\}$ and $c^{\mathcal{D}} = c$ for any $c \in D$.

Boolean algebras

The theory of *Boolean algebras* has the language $L = \langle -, \wedge, \vee, 0, 1 \rangle$ with equality and the following axioms.

$x \wedge (y \wedge z) = (x \wedge y) \wedge z$	(associativity of \wedge)
$x \vee (y \vee z) = (x \vee y) \vee z$	(associativity of \vee)
$x \wedge y = y \wedge x$	(commutativity of \wedge)
$x \vee y = y \vee x$	(commutativity of \vee)
$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$	(distributivity of \wedge over \vee)
$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$	(distributivity of \vee over \wedge)
$x \wedge (x \vee y) = x, \quad x \vee (x \wedge y) = x$	(absorption)
$x \vee (-x) = 1, \quad x \wedge (-x) = 0$	(complementation)
$0 \neq 1$	(non-triviality)

The smallest model is $\underline{2} = \langle \{0, 1\}, -, \wedge_1, \vee_1, 0, 1 \rangle$. Finite Boolean algebras are (up to isomorphism) $\langle \{0, 1\}^n, -, \wedge_n, \vee_n, 0_n, 1_n \rangle$ for $n \in \mathbb{N}^+$, where the operations (on binary n -tuples) are the coordinate-wise operations of $\underline{2}$.

Relations of propositional and predicate logic

- Propositional formulas over connectives \neg, \wedge, \vee (eventually with \top, \perp) can be viewed as **Boolean terms**. Then the truth value of φ in a given assignment is the value of the term in the Boolean algebra $\underline{2}$.
- **Algebra of propositions** over \mathbb{P} is a Boolean algebra (also for \mathbb{P} infinite).
- If we represent atomic subformulas in an **open** formula φ (without equality) with propositional letters, we obtain a proposition that is valid if and only if φ is valid.
- Propositional logic can be introduced as a **fragment** of predicate logic using **nullary** relation symbols (*syntax*) and nullary relations (*semantics*) since $A^0 = \{\emptyset\} = 1$, so $R^A \subseteq A^0$ is either $R^A = \emptyset = 0$ or $R^A = \{\emptyset\} = 1$.

Tableau method in propositional logic - a review

- A **tableau** is a binary tree that represents a search for a *counterexample*.
- Nodes are labeled by **entries**, i.e. formulas with a **sign** T / F that represents an assumption that the formula is **true / false** in some model.
- If this assumption is correct, then it is correct also for all the entries in some branch below that came from this entry.
- A branch is **contradictory** (it fails) if it contains $T\psi, F\psi$ for some ψ .
- A **proof** of formula φ is a **contradictory** tableau with root $F\varphi$, i.e. a tableau in which every branch is contradictory. If φ has a proof, it is valid.
- If a counterexample exists, there will be a branch in a **finished** tableau that **provides** us with this counterexample, but this branch can be infinite.
- We can construct a **systematic tableau** that is always finished.
- If φ is valid, the systematic tableau for φ is contradictory, i.e. it is a proof of φ ; and in this case, it is also **finite**.

Tableau method in predicate logic - what is different

- Formulas in entries will always be **sentences** (closed formulas), i.e. formulas without free variables.
- We add **new atomic tableaux** for quantifiers.
- In these tableaux we substitute **ground terms** for quantified variables following certain rules.
- We extend the language by **new (auxiliary) constant symbols** (countably many) to represent “*witnesses*” of entries $T(\exists x)\varphi(x)$ and $F(\forall x)\varphi(x)$.
- In a **finished** noncontradictory branch containing an entry $T(\forall x)\varphi(x)$ or $F(\exists x)\varphi(x)$ we have **instances** $T\varphi(x/t)$ resp. $F\varphi(x/t)$ for every ground term t (of the extended language).

Assumptions

- 1) The formula φ that we want to prove (or refute) is a **sentence**. If not, we can replace φ with its **universal closure** φ' , since for every theory T ,

$$T \models \varphi \quad \text{if and only if} \quad T \models \varphi'.$$

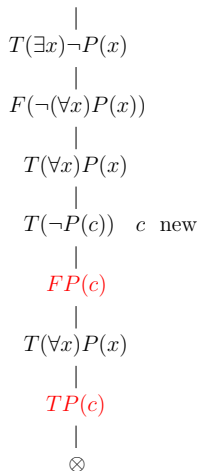
- 2) We prove from a theory in a **closed form**, i.e. every axiom is a sentence. By replacing every axiom ψ with its universal closure ψ' we obtain an **equivalent** theory since for every structure \mathcal{A} (of the given language L),

$$\mathcal{A} \models \psi \quad \text{if and only if} \quad \mathcal{A} \models \psi'.$$

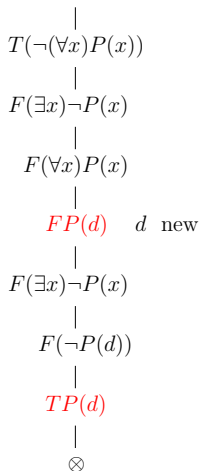
- 3) The language L is **countable**. Then every theory of L is countable. We denote by L_C the extension of L by new constant symbols c_0, c_1, \dots (countably many). Then there are countably many ground terms of L_C . Let t_i denote the i -th ground term (in some fixed **enumeration**).
- 4) First, we assume that the language is **without equality**.

Tableaux in predicate logic - examples

$$F((\exists x)\neg P(x) \rightarrow \neg(\forall x)P(x))$$



$$F(\neg(\forall x)P(x) \rightarrow (\exists x)\neg P(x))$$



Atomic tableaux - previous

An *atomic tableau* is one of the following trees (labeled by entries), where α is any atomic sentence and φ, ψ are any sentences, all of language L_C .

$T\alpha$	$F\alpha$	$ \begin{array}{c} T(\varphi \wedge \psi) \\ \\ T\varphi \\ \\ T\psi \end{array} $	$ \begin{array}{c} F(\varphi \wedge \psi) \\ / \quad \backslash \\ F\varphi \quad F\psi \end{array} $	$ \begin{array}{c} T(\varphi \vee \psi) \\ / \quad \backslash \\ T\varphi \quad T\psi \end{array} $	$ \begin{array}{c} F(\varphi \vee \psi) \\ \\ F\varphi \\ \\ F\psi \end{array} $
$ \begin{array}{c} T(\neg\varphi) \\ \\ F\varphi \end{array} $	$ \begin{array}{c} F(\neg\varphi) \\ \\ T\varphi \end{array} $	$ \begin{array}{c} T(\varphi \rightarrow \psi) \\ / \quad \backslash \\ F\varphi \quad T\psi \end{array} $	$ \begin{array}{c} F(\varphi \rightarrow \psi) \\ \\ T\varphi \\ \\ F\psi \end{array} $	$ \begin{array}{c} T(\varphi \leftrightarrow \psi) \\ / \quad \backslash \\ T\varphi \quad F\varphi \\ \quad \quad \\ T\psi \quad F\psi \end{array} $	$ \begin{array}{c} F(\varphi \leftrightarrow \psi) \\ / \quad \backslash \\ T\varphi \quad F\varphi \\ \quad \quad \\ F\psi \quad T\psi \end{array} $

Atomic tableaux - new

Atomic tableaux are also the following trees (labeled by entries), where φ is any formula of the language L_C with a free variable x , t is any ground term of L_C and c is a **new** constant symbol from $L_C \setminus L$.

#	*	*	#
$T(\forall x)\varphi(x)$	$F(\forall x)\varphi(x)$	$T(\exists x)\varphi(x)$	$F(\exists x)\varphi(x)$
$T\varphi(x/t)$	$F\varphi(x/c)$	$T\varphi(x/c)$	$F\varphi(x/t)$
for any ground term t of L_C	for a <i>new</i> constant c	for a <i>new</i> constant c	for any ground term t of L_C

Remark The constant symbol c represents a “witness” of the entry $T(\exists x)\varphi(x)$ or $F(\forall x)\varphi(x)$. Since we need that no prior demands are put on c , we specify (in the definition of a tableau) which constant symbols c may be used.

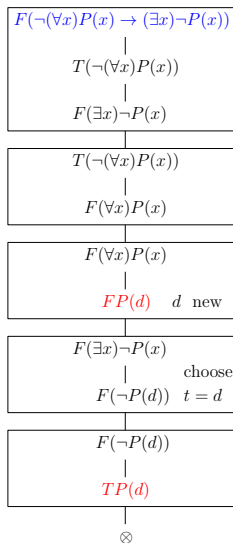
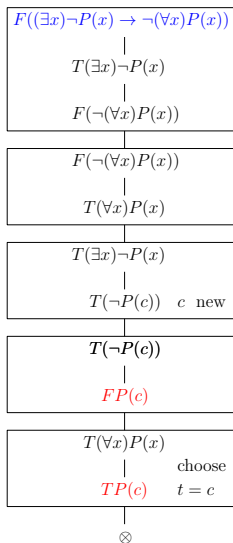
Tableau

A **finite tableau** from a theory T is a binary tree labeled with entries described

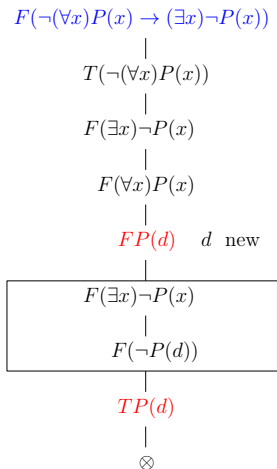
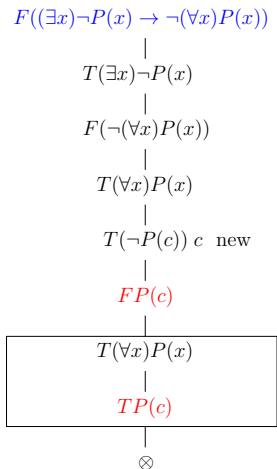
- (i) every atomic tableau is a finite tableau from T , whereas in case (*) we may use any constant symbol $c \in L_C \setminus L$,
- (ii) if P is an entry on a branch V in a finite tableau from T , then by adjoining the atomic tableau for P at the **end of branch** V we obtain (again) a finite tableau from T , whereas in case (*) we may use only a constant symbol $c \in L_C \setminus L$ that **does not appear** on V ,
- (iii) if V is a branch in a finite tableau from T and $\varphi \in T$, then by adjoining $T\varphi$ at the end of branch V we obtain (again) a finite tableau from T .
- (iv) every finite tableau from T is formed by **finitely** many steps (i), (ii), (iii).

A **tableau** from T is a sequence $\tau_0, \tau_1, \dots, \tau_n, \dots$ of finite tableaux from T such that τ_{n+1} is formed from τ_n by (ii) or (iii), formally $\tau = \cup \tau_n$.

Construction of tableaux



Convention



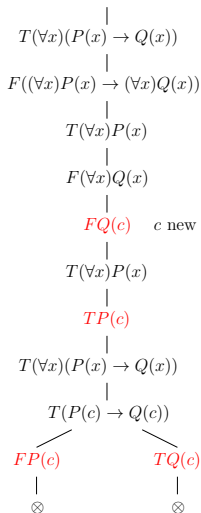
We will not write the entry that is expanded again on the branch, except in cases when the entry is in the form of $T(\forall x)\varphi(x)$ or $F(\exists x)\varphi(x)$.

Tableau proof

- A branch V in a tableau τ is *contradictory* if it contains entries $T\varphi$ and $F\varphi$ for some sentence φ , otherwise V is *noncontradictory*.
- A tableau τ is *contradictory* if every branch in τ is contradictory.
- A *tableau proof* (*proof by tableau*) of a sentence φ from a theory T is a *contradictory tableau* from T with $F\varphi$ in the root.
- A sentence φ is *(tableau) provable* from T , denoted by $T \vdash \varphi$, if it has a tableau proof from T .
- A *refutation* of a sentence φ by *tableau* from a theory T is a *contradictory tableau* from T with the root entry $T\varphi$.
- A sentence φ is *(tableau) refutable* from T if it has a refutation by tableau from T , i.e. $T \vdash \neg\varphi$.

Examples

$$F((\forall x)(P(x) \rightarrow Q(x)) \rightarrow ((\forall x)P(x) \rightarrow (\forall x)Q(x)))$$



$$F((\forall x)(\varphi(x) \wedge \psi(x)) \leftrightarrow ((\forall x)\varphi(x) \wedge (\forall x)\psi(x)))$$

