Propositional and Predicate Logic - IX

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Finished tableau

A finished noncontradictory branch should provide us with a counterexample.

An occurrence of an entry P in a node v of a tableau τ is *i-th* if v has exactly $i-1$ predecessors labeled by P; and is *reduced* on a branch V through v if

$a)$ P is neither in form of $T(\forall x)\varphi(x)$ nor $F(\exists x)\varphi(x)$ and P occurs on V as a root of an atomic tableau, i.e. it was already expanded on V, or

$b)$ P is in form of $T(\forall x)\varphi(x)$ or $F(\exists x)\varphi(x)$, P has an $(i+1)$-th occurrence on V, and V contains an entry $T\varphi(x/t_i)$ resp. $F\varphi(x/t_i)$ where t_i is the i-th ground term (of the language L_C).

Let V be a branch in a tableau τ from a theory T. We say that

- V is *finished* if it is contradictory, or every occurrence of an entry on V is reduced on V and, moreover, V contains $T\varphi$ for every $\varphi \in T$,
- τ is *finished* if every branch in τ is finished.

Systematic tableau - construction

Let R be an entry and $T = \{\varphi_0, \varphi_1, \dots\}$ be a (possibly infinite) theory.

(1) We take the atomic tableau for R as τ_0. In case $(*)$ we choose any $c \in L_C \setminus L$, in case $(\sharp)$ we take t_1 for t. Till possible, proceed as follows.

(2) Let v be the leftmost node in the smallest level as possible in tableau τ_n containing an occurrence of an entry P that is not reduced on some noncontradictory branch through v. (If v does not exist, we take $\tau_n' = \tau_n$.)

$(3a)$ If P is neither $T(\forall x)\varphi(x)$ nor $F(\exists x)\varphi(x)$, let τ_n' be the tableau obtained from τ_n by adjoining the atomic tableau for P to every noncontradictory branch through v. In case $(*)$ we choose c_i for the smallest possible i.

$(3b)$ If P is $T(\forall x)\varphi(x)$ or $F(\exists x)\varphi(x)$ and it has i-th occurrence in v, let τ_n' be the tableau obtained from τ_n by adjoining atomic tableau for P to every noncontradictory branch through v, where we take the term t_i for t.

(4) Let τ_{n+1} be the tableau obtained from τ_n' by adjoining $T\varphi_n$ to every noncontradictory branch that does not contain $T\varphi_n$ yet. (If φ_n does not exist, we take $\tau_{n+1} = \tau_n'$.)

The *systematic tableau* for R from T is the result $\tau = \cup \tau_n$ of this construction.

Systematic tableau - an example

$$T((\exists y)(\neg R(y,y) \vee P(y,y)) \wedge (\forall x)R(x,x))$$

$$T(\exists y)(\neg R(y,y) \vee P(y,y))$$

$$T(\forall x)R(x,x)$$

$$T(\neg R(c_0,c_0) \vee P(c_0,c_0)) \qquad c_0 \;\; \text{new}$$

$$T(\forall x)R(x,x)$$

$$TR(c_0,c_0) \qquad (\text{assuming that } t_1 = c_0)$$

$$\begin{array}{cc}
T(\neg R(c_0,c_0)) & TP(c_0,c_0) \\
| & | \\
T(\forall x)R(x,x) & T(\forall x)R(x,x) \\
| & | \\
TR(t_2,t_2) & TR(t_2,t_2) \\
| & | \\
FR(c_0,c_0) & T(\forall x)R(x,x) \\
| & | \\
\otimes & TR(t_3,t_3) \\
 & \vdots
\end{array}$$

Systematic tableau - being finished

Proposition *Every systematic tableau is finished.*

Proof Let $\tau = \cup\tau_n$ be a systematic tableau from $T = \{\varphi_0, \varphi_1, \dots\}$ with root R and let P be an entry in a node v of the tableau τ.

- There are only finitely many entries in τ in levels up to the level of v.
- If the occurrence of P in v was unreduced on some noncontradictory branch in τ, it would be found in some step (2) and reduced by $(3a)$, $(3b)$.
- By step (4) every $\varphi_n \in T$ will be (no later than) in τ_{n+1} on every noncontradictory branch.
- Hence the systematic tableau τ has all branches finished. $\square$

Proposition *If a systematic tableau τ is a proof (from a theory T), it is finite.*

Proof Suppose that τ is infinite. Then by König's lemma, τ contains an infinite branch. This branch is noncontradictory since in the construction only noncontradictory branches are prolonged. But this contradicts the assumption that τ is a contradictory tableau. $\square$

Equality

Axioms of equality for a language L with equality are

(i) $x = x$

(ii) $x_1 = y_1 \ \land \ \cdots \ \land \ x_n = y_n \ \ \rightarrow \ \ f(x_1, \ldots, x_n) = f(y_1, \ldots, y_n)$
for each n-ary function symbol f of the language L.

(iii) $x_1 = y_1 \ \land \ \cdots \ \land \ x_n = y_n \ \ \rightarrow \ \ (R(x_1, \ldots, x_n) \rightarrow R(y_1, \ldots, y_n))$
for each n-ary relation symbol R of the language L including $=$.

A *tableau proof* from a theory T in a language L *with equality* is a tableau proof from T^* where T^* denotes the extension of T by adding axioms of equality for L *(resp. their universal closures).*

Remark *In context of logic programming the equality often has other meaning than in mathematics (identity). For example in Prolog, $t_1 = t_2$ means that t_1 and t_2 are unifiable.*

Congruence and quotient structure

Let $\sim$ be an equivalence on A, $f : A^n \to A$, and $R \subseteq A^n$ for $n \in \mathbb{N}$. Then $\sim$ is

- a *congruence for the function* f if for every $x_1, \dots, x_n, y_1, \dots, y_n \in A$
$$x_1 \sim y_1 \ \wedge \ \cdots \ \wedge \ x_n \sim y_n \quad \Rightarrow \quad f(x_1, \dots, x_n) \sim f(y_1, \dots, y_n),$$
- a *congruence for the relation* R if for every $x_1, \dots, x_n, y_1, \dots, y_n \in A$
$$x_1 \sim y_1 \ \wedge \ \cdots \ \wedge \ x_n \sim y_n \quad \Rightarrow \quad (R(x_1, \dots, x_n) \Leftrightarrow R(y_1, \dots, y_n)).$$

Let an equivalence $\sim$ on A be a congruence for every function and relation in a structure $\mathcal{A} = \langle A, \mathcal{F}^A, \mathcal{R}^A \rangle$ of language $L = \langle \mathcal{F}, \mathcal{R} \rangle$. Then the *quotient* (*structure*) of $\mathcal{A}$ by $\sim$ is the structure $\mathcal{A}/\sim = \langle A/\sim, \mathcal{F}^{A/\sim}, \mathcal{R}^{A/\sim} \rangle$ where

$$f^{A/\sim}([x_1]_\sim, \dots, [x_n]_\sim) = [f^A(x_1, \dots, x_n)]_\sim$$
$$R^{A/\sim}([x_1]_\sim, \dots, [x_n]_\sim) \Leftrightarrow R^A(x_1, \dots, x_n)$$

for each $f \in \mathcal{F}$, $R \in \mathcal{R}$, and $x_1, \dots, x_n \in A$, i.e. the functions and relations are defined from $\mathcal{A}$ using representatives.

Example: $\underline{\mathbb{Z}}_p$ *is the quotient of* $\underline{\mathbb{Z}} = \langle \mathbb{Z}, +, -, 0 \rangle$ *by the congruence modulo* p.

Role of axioms of equality

Let $\mathcal{A}$ be a structure of a language L in which the equality is interpreted as a relation $=^A$ satisfying the axioms of equality for L, i.e. not necessarily the identity relation.

1) From axioms (i) and (iii) it follows that the relation $=^A$ is an equivalence.
2) Axioms (ii) and (iii) express that the relation $=^A$ is a congruence for every function and relation in $\mathcal{A}$.
3) If $\mathcal{A} \models T^*$ then also $(\mathcal{A}/{=^A}) \models T^*$ where $\mathcal{A}/{=^A}$ is the quotient of $\mathcal{A}$ by $=^A$. Moreover, the equality is interpreted in $\mathcal{A}/{=^A}$ as the identity relation.

On the other hand, in every model in which the equality is interpreted as the identity relation, all axioms of equality evidently hold.

Soundness

We say that a model $\mathcal{A}$ *agrees* with an entry P, if P is $T\varphi$ and $\mathcal{A} \models \varphi$ or if P is $F\varphi$ and $\mathcal{A} \models \neg\varphi$, i.e. $\mathcal{A} \not\models \varphi$. Moreover, $\mathcal{A}$ *agrees* with a branch V if $\mathcal{A}$ agrees with every entry on V.

Lemma *Let $\mathcal{A}$ be a model of a theory T of a language L that agrees with the root entry R in a tableau $\tau = \cup\tau_n$ from T. Then $\mathcal{A}$ can be expanded to the language L_C so that it agrees with some branch V in τ.*

Remark *It suffices to expand $\mathcal{A}$ only by constants c^A such that $c \in L_C \setminus L$ occurs on V, other constants may be defined arbitrarily.*

Proof By induction on n we find a branch V_n in τ_n and an expansion $\mathcal{A}_n$ of $\mathcal{A}$ by constants c^A for all $c \in L_C \setminus L$ on V_n s.t. $\mathcal{A}_n$ agrees with V_n and $V_{n-1} \subseteq V_n$.

Assume we have a branch V_n in τ_n and an expansion $\mathcal{A}_n$ that agrees with V_n.

- If τ_{n+1} is formed from τ_n without extending the branch V_n, we take $V_{n+1} = V_n$ and $\mathcal{A}_{n+1} = \mathcal{A}_n$.
- If τ_{n+1} is formed from τ_n by appending $T\varphi$ to V_n for some $\varphi \in T$, let V_{n+1} be this branch and $\mathcal{A}_{n+1} = \mathcal{A}_n$. Since $\mathcal{A} \models \varphi$, $\mathcal{A}_{n+1}$ agrees with V_{n+1}.

Soundness - proof (cont.)

- Otherwise τ_{n+1} is formed from τ_n by appending an atomic tableau to V_n for some entry P on V_n. By induction we know that $\mathcal{A}_n$ agrees with P.

(i) If P is formed by a logical connective, we take $\mathcal{A}_{n+1} = \mathcal{A}_n$ and verify that V_n can always be extended to a branch V_{n+1} agreeing with $\mathcal{A}_{n+1}$.

(ii) If P is in form $T(\forall x)\varphi(x)$, let V_{n+1} be the (unique) extension of V_n to a branch in τ_{n+1}, i.e. by the entry $T\varphi(x/t)$. Let $\mathcal{A}_{n+1}$ be any expansion by new constants from t. Since $\mathcal{A}_n \models (\forall x)\varphi(x)$, we have $\mathcal{A}_{n+1} \models \varphi(x/t)$. Analogously for P in form $F(\exists x)\varphi(x)$.

(iii) If P is in form $T(\exists x)\varphi(x)$, let V_{n+1} be the (unique) extension of V_n to a branch in τ_{n+1}, i.e. by the entry $T\varphi(x/c)$. Since $\mathcal{A}_n \models (\exists x)\varphi(x)$, there is some $a \in A$ with $\mathcal{A}_n \models \varphi(x)[e(x/a)]$ for every assignment e. Let $\mathcal{A}_{n+1}$ be the expansion of $\mathcal{A}_n$ by a new constant $c^A = a$. Then $\mathcal{A}_{n+1} \models \varphi(x/c)$. Analogously for P in form $F(\forall x)\varphi(x)$.

The base step for $n = 0$ follows from similar analysis of atomic tableaux for the root entry R applying the assumption that $\mathcal{A}$ agrees with R. $\square$

Theorem on soundness

We will show that the tableau method in predicate logic is sound.

Theorem *For every theory T and sentence φ, if φ is tableau provable from T, then φ is valid in T, i.e. $T \vdash \varphi \;\Rightarrow\; T \models \varphi$.*

Proof

- Let φ be tableau provable from a theory T, i.e. there is a contradictory tableau τ from T with the root entry $F\varphi$.
- Suppose for a contradiction that φ is not valid in T, i.e. there exists a model $\mathcal{A}$ of the theory T in which φ is not true (a counterexample).
- Since $\mathcal{A}$ agrees with the root entry $F\varphi$, by the previous lemma, $\mathcal{A}$ can be expanded to the language L_C so that it agrees with some branch in τ.
- But this is impossible, since every branch of τ is contradictory, i.e. it contains a pair of entries $T\psi$, $F\psi$ for some sentence ψ. $\square$

The canonical model

From a noncontradictory branch V of a finished tableau we build a model that agrees with V. We build it on available (syntactical) objects - ground terms.

Let V be a noncontradictory branch of a finished tableau from a theory T of a language $L = \langle \mathcal{F}, \mathcal{R} \rangle$. The *canonical model* from V is the L_C-structure $\mathcal{A} = \langle A, \mathcal{F}^A, \mathcal{R}^A \rangle$ where

(1) A is the set of all ground terms of the language L_C,

(2) $f^A(t_{i_1}, \ldots, t_{i_n}) = f(t_{i_1}, \ldots, t_{i_n})$
for every n-ary function symbol $f \in \mathcal{F} \cup (L_C \setminus L)$ and $t_{i_1}, \ldots, t_{i_n} \in A$.

(3) $R^A(t_{i_1}, \ldots, t_{i_n}) \Leftrightarrow T\, R(t_{i_1}, \ldots, t_{i_n})$ is an entry on V
for every n-ary relation symbol $R \in \mathcal{R}$ or equality and $t_{i_1}, \ldots, t_{i_n} \in A$.

Remark *The expression $f(t_{i_1}, \ldots, t_{i_n})$ on the right side of (2) is a ground term of L_C, i.e. an element of A. Informally, to indicate that it is a syntactical object*

$$f^A(t_{i_1}, \ldots, t_{i_n}) = \text{“}f(t_{i_1}, \ldots, t_{i_n})\text{”}$$

The canonical model - an example

Let $T = \{(\forall x)R(f(x))\}$ be a theory of a language $L = \langle R, f, d\rangle$. The systematic tableau for $F\neg R(d)$ from T contains a single branch V, which is noncontradictory.

The canonical model $\mathcal{A} = \langle A, R^A, f^A, d^A, c_i^A\rangle_{i\in\mathbb{N}}$ from V is for language L_C and

$$A = \{d, f(d), f(f(d)), \dots, c_0, f(c_0), f(f(c_0)), \dots, c_1, f(c_1), f(f(c_1)), \dots\},$$

$$d^A = d, \quad c_i^A = c_i \text{ for } i \in \mathbb{N},$$

$$f^A(d) = \text{“}f(d)\text{”}, \; f^A(f(d)) = \text{“}f(f(d))\text{”}, \; f^A(f(f(d))) = \text{“}f(f(f(d)))\text{”}, \dots$$

$$R^A = \{d, f(d), f(f(d)), \dots, f(c_0), f(f(c_0)), \dots, f(c_1), f(f(c_1)), \dots\}.$$

The reduct of $\mathcal{A}$ to the language L is $\mathcal{A}' = \langle A, R^A, f^A, d^A\rangle$.

The canonical model with equality

If L is with equality, T^* is the extension of T by the axioms of equality for L.

If we require that the equality is interpreted as the identity, we have to take the quotient of the canonical model $\mathcal{A}$ by the congruence $=^A$.

By (3), for the relation $=^A$ in $\mathcal{A}$ from V it holds that for every $s, t \in A$,

$$s =^A t \iff T(s = t) \text{ is an entry on } V.$$

Since V is finished and contains the axioms of equality, the relation $=^A$ is a congruence for all functions and relations in $\mathcal{A}$.

The *canonical model with equality* from V is the quotient $\mathcal{A}/{=^A}$.

Observation *For every formula φ,*

$$\mathcal{A} \models \varphi \iff (\mathcal{A}/{=^A}) \models \varphi,$$

where $=$ is interpreted in $\mathcal{A}$ by the relation $=^A$, while in $\mathcal{A}/{=^A}$ by the identity.

Remark *$\mathcal{A}$ is a countably infinite model, but $\mathcal{A}/{=^A}$ can be finite.*

The canonical model with equality - an example

Let $T = \{(\forall x)R(f(x)),\ (\forall x)(x = f(f(x)))\}$ be of $L = \langle R, f, d\rangle$ with equality.
The systematic tableau for $F\neg R(d)$ from T^* contains a noncontradictory V.

In the canonical model $\mathcal{A} = \langle A, R^A, =^A, f^A, d^A, c_i^A\rangle_{i\in\mathbb{N}}$ from V we have that

$$s =^A t \quad \Leftrightarrow \quad t = f(\cdots(f(s)\cdots) \text{ or } s = f(\cdots(f(t)\cdots),$$

where f is applied $2i$-times for some $i \in \mathbb{N}$.

The canonical model with equality from V is
$\mathcal{B} = (\mathcal{A}/=^A) = \langle A/=^A, R^B, f^B, d^B, c_i^B\rangle_{i\in\mathbb{N}}$ where

$$\begin{aligned}
(A/=^A) &= \{[d]_{=^A}, [f(d)]_{=^A}, [c_0]_{=^A}, [f(c_0)]_{=^A}, [c_1]_{=^A}, [f(c_1)]_{=^A}, \ldots\},\\
d^B &= [d]_{=^A}, \quad c_i^B = [c_i]_{=^A} \text{ for } i \in \mathbb{N},\\
f^B([d]_{=^A}) &= [f(d)]_{=^A},\ f^B([f(d)]_{=^A}) = [f(f(d))]_{=^A} = [d]_{=^A},\ \ldots\\
R^B &= (A/=^A).
\end{aligned}$$

The reduct of $\mathcal{B}$ to the language L is $\mathcal{B}' = \langle A/=^A, R^B, f^B, d^B\rangle$.

Completeness

Lemma *The canonical model $\mathcal{A}$ from a noncontr. finished V agrees with V.*

Proof By induction on the structure of a sentence in an entry on V.

- For atomic φ, if $T\varphi$ is on V, then $\mathcal{A} \models \varphi$ by (3). If $F\varphi$ is on V, then $T\varphi$ is not on V since V is noncontradictory, so $\mathcal{A} \models \neg\varphi$ by (3).
- If $T(\varphi \wedge \psi)$ is on V, then $T\varphi$ and $T\psi$ are on V since V is finished. By induction, $\mathcal{A} \models \varphi$ and $\mathcal{A} \models \psi$, and thus $\mathcal{A} \models \varphi \wedge \psi$.
- If $F(\varphi \wedge \psi)$ is on V, then $F\varphi$ or $F\psi$ is on V since V is finished. By induction, $\mathcal{A} \models \neg\varphi$ or $\mathcal{A} \models \neg\psi$, and thus $\mathcal{A} \models \neg(\varphi \wedge \psi)$.
- For other connectives similarly as in previous two cases.
- If $T(\forall x)\varphi(x)$ is on V, then $T\varphi(x/t)$ is on V for every $t \in A$ since V is finished. By induction, $\mathcal{A} \models \varphi(x/t)$ for every $t \in A$, and thus $\mathcal{A} \models (\forall x)\varphi(x)$. Similarly for $F(\exists x)\varphi(x)$ on V.
- If $T(\exists x)\varphi(x)$ is on V, then $T\varphi(x/c)$ is on V for some $c \in A$ since V is finished. By induction, $\mathcal{A} \models \varphi(x/c)$, and thus $\mathcal{A} \models (\exists x)\varphi(x)$. Similarly for $F(\forall x)\varphi(x)$ on V. $\square$

Theorem on completeness

We will show that the tableau method in predicate logic is complete.

Theorem *For every theory T and sentence φ, if φ is valid in T, then φ is tableau provable from T, i.e. $T \models \varphi \;\Rightarrow\; T \vdash \varphi$.*

Proof Let φ be valid in T. We will show that an arbitrary finished tableau (e.g. systematic) τ from a theory T with the root entry $F\varphi$ is contradictory.

- If not, then there is some noncontradictory branch V in τ.
- By the previous lemma, there is a structure $\mathcal{A}$ for L_C that agrees with V, in particular with the root entry $F\varphi$, i.e. $\mathcal{A} \models \neg\varphi$.
- Let $\mathcal{A}'$ be the reduct of $\mathcal{A}$ to the language L. Then $\mathcal{A}' \models \neg\varphi$.
- Since V is finished, it contains $T\psi$ for every $\psi \in T$.
- Thus $\mathcal{A}'$ is a model of T (as $\mathcal{A}'$ agrees with $T\psi$ for every $\psi \in T$).
- But this contradicts the assumption that φ is valid in T.

Therefore the tableau τ is a proof of φ from T. $\square$

Properties of theories

We introduce syntactic variants of previous semantical definitions.

Let T be a theory of a language L. If a sentence φ is provable from T, we say that φ is a *theorem* of T. The set of theorems of T is denoted by

$$\mathrm{Thm}^L(T) = \{\varphi \in \mathrm{Fm}_L \mid T \vdash \varphi\}.$$

We say that a theory T is

- *inconsistent* if $T \vdash \perp$, otherwise T is *consistent*,
- *complete* if it is consistent and every sentence is provable or refutable from T, i.e. $T \vdash \varphi$ or $T \vdash \neg\varphi$.
- an *extension* of a theory T' of L' if $L' \subseteq L$ and $\mathrm{Thm}^{L'}(T') \subseteq \mathrm{Thm}^L(T)$, we say that an extension T of a theory T' is *simple* if $L = L'$; and *conservative* if $\mathrm{Thm}^{L'}(T') = \mathrm{Thm}^L(T) \cap \mathrm{Fm}_{\mathrm{L}'}$,
- *equivalent* with a theory T' if T is an extension of T' and vice-versa.

Corollaries

From the soundness and completeness of the tableau method it follows that these syntactic definitions agree with their semantic variants.

Corollary *For every theory T and sentences φ, ψ of a language L,*

- $T \vdash \varphi$ *if and only if* $T \models \varphi$,
- $\mathrm{Thm}^L(T) = \theta^L(T)$,
- T *is inconsistent if and only if* T *is unsatisfiable, i.e. it has no model,*
- T *is complete if and only if* T *is semantically complete, i.e. it has a single model, up to elementarily equivalence,*
- $T, \varphi \vdash \psi$ *if and only if* $T \vdash \varphi \rightarrow \psi$ *(Deduction theorem).*

Remark Deduction theorem can be proved directly by transformations of tableaux.

Existence of a countable model and compactness

Theorem *Every consistent theory T of a countable language L without equality has a countably infinite model.*

Proof Let τ be the systematic tableau from T with $F\bot$ in the root. Since τ is finished and contains a noncontradictory branch V as $\bot$ is not provable from T, there exists a canonical model $\mathcal{A}$ from V. Since $\mathcal{A}$ agrees with V, its reduct to the language L is a desired countably infinite model of T. □

Remark This is a weak version of so called Löwenheim-Skolem theorem. In a countable language with equality the canonical model with equality is countable (i.e. finite or countably infinite).

Theorem *A theory T has a model iff every finite subset of T has a model.*

Proof The implication from left to right is obvious. If T has no model, then it is inconsistent, i.e. $\bot$ is provable by a systematic tableau τ from T. Since τ is finite, $\bot$ is provable from some finite $T' \subseteq T$, i.e. T' has no model. □

Non-standard model of natural numbers

Let $\underline{\mathbb{N}} = \langle \mathbb{N}, S, +, \cdot, 0, \leq \rangle$ be the standard model of natural numbers.

Let $\mathrm{Th}(\underline{\mathbb{N}})$ denote the set of all sentences that are valid in $\underline{\mathbb{N}}$. For $n \in \mathbb{N}$ let $\underline{n}$ denote the term $S(S(\cdots(S(0))\cdots))$, so called the *n-th numeral*, where S is applied n-times.

Consider the following theory T where c is a new constant symbol.

$$T = \mathrm{Th}(\underline{\mathbb{N}}) \cup \{\underline{n} < c \mid n \in \mathbb{N}\}$$

Observation Every finite subset of T has a model.

Thus by the compactness theorem, T has a model $\mathcal{A}$. It is a non-standard model of natural numbers. Every sentence from $\mathrm{Th}(\underline{\mathbb{N}})$ is valid in $\mathcal{A}$ but it contains an element c^A that is greater then every $n \in \mathbb{N}$ (i.e. the value of the term $\underline{n}$ in $\mathcal{A}$).