# Propositional and Predicate Logic - Appendix


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# Set-theoretical notions

All notions are introduced within a set theory using only the membership predicate and equality (and means of logic).

- A property of sets $\varphi(x)$ defines a *class* $\{x \mid \varphi(x)\}$. A class that is not a set is called a *proper* class, eg. $\{x \mid x = x\}$,
- $x \notin y$, $x \neq y$ are shortcuts for $\neg(x \in y)$, $\neg(x = y)$,
- $\{x_0, \ldots, x_{n-1}\}$ denotes the set containing exactly $x_0, \ldots, x_{n-1}$, $\{x\}$ is called a *singleton*, $\{x, y\}$ is called an *unordered pair*,
- $\emptyset$, $\cup$, $\cap$, $\setminus$, $\triangle$ stand for *empty set*, *union*, *intersection*, *difference*, *symmetric difference* of sets, e.g.
$$x \triangle y = (x \setminus y) \cup (y \setminus x) = \{z \mid (z \in x \wedge z \notin y) \vee (z \notin x \wedge z \in y)\}$$
- $x, y$ are *disjoint* if $x \cap y = \emptyset$, we denote by $x \subseteq y$ that $x$ is a *subset* of $y$,
- the *power set* of $x$ is $\mathcal{P}(x) = \{y \mid y \subseteq x\}$,
- the *union* of $x$ is $\bigcup x = \{z \mid \exists y(z \in y \wedge y \in x)\}$,
- a *cover* of a set $x$ is a set $y \subseteq \mathcal{P}(x) \setminus \{\emptyset\}$ with $\bigcup y = x$. If, moreover, all sets in $y$ are mutually disjoint, then $y$ is a *partition* of $x$.

# Relations

- An *ordered pair* is $(x, y) = \{x, \{x, y\}\}$, so $(x, x) = \{x, \{x\}\}$,
  an *ordered $n$-tuple* is $(x_0, \dots, x_{n-1}) = ((x_0, \dots, x_{n-2}), x_{n-1})$ for $n > 2$,
- the *Cartesian product* of $a$ and $b$ is $a \times b = \{(x, y) \mid x \in a, y \in b\}$,
  the *Cartesian power* of $x$ is $x^0 = \{\emptyset\}$, $x^1 = x$, $x^n = x^{n-1} \times x$ for $n > 1$,
- the *disjoint union* of $x$ and $y$ is $x \uplus y = (\{\emptyset\} \times x) \cup (\{\{\emptyset\}\} \times y)$,
- a *relation* is a set $R$ of ordered pairs, instead of $(x, y) \in R$ we usually write $R(x, y)$ or $x\,R\,y$,
  the *domain* of $R$ is $\mathrm{dom}(R) = \{x \mid \exists y\ (x, y) \in R\}$,
  the *range* of $R$ is $\mathrm{rng}(R) = \{y \mid \exists x\ (x, y) \in R\}$,
  the *extension* of $x$ in $R$ is $R[x] = \{y \mid (x, y) \in R\}$,
  the *inverse relation* to $R$ is $R^{-1} = \{(y, x) \mid (x, y) \in R\}$,
  the *restriction* of $R$ to the set $z$ is $R \restriction z = \{(x, y) \in R \mid x \in z\}$,
- the *composition* of relations $R$ and $S$ is the relation
  $$R \circ S = \{(x, z) \mid \exists y\ ((x, y) \in R \wedge (y, z) \in S)\},$$
- the *identity* on a set $z$ is the relation $\mathrm{Id}_z = \{(x, x) \mid x \in z\}$.

# Equivalences

- A relation $R$ on $X$ is an *equivalence* if for every $x, y, z \in X$

$$R(x,x) \quad \text{(reflexivity)}$$
$$R(x,y) \rightarrow R(y,x) \quad \text{(symmetry)}$$
$$R(x,y) \wedge R(y,z) \rightarrow R(x,z) \quad \text{(transitivity)}$$

- $R[x]$ is called the *equivalence class* of $x$ in $R$, denoted also $[x]_R$.
- $X/R = \{R[x] \mid x \in X\}$ is the *quotient set* of $X$ by $R$.
- It holds that $X/R$ is a partition of $X$ since the equivalence classes are mutually disjoint and cover $X$.
- On the other hand, a partition $S$ of $X$ determines the equivalence (on $X$)

$$\{(x,y) \mid x \in z, y \in z \text{ for some } z \in S\}.$$

# Orders

Let $\leq$ be a relation on a set $X$. We say that $\leq$ is

- a *partial order* (of the set $X$) if for every $x, y, z \in X$

$$x \leq x \qquad \text{(reflexivity)}$$

$$x \leq y \ \wedge\ y \leq x \ \rightarrow\ x = y \qquad \text{(antisymmetry)}$$

$$x \leq y \ \wedge\ y \leq z \ \rightarrow\ x \leq z \qquad \text{(transitivity)}$$

- a *linear* (*total*) *order* if, moreover, for every $x, y \in X$

$$x \leq y \ \vee\ y \leq x \qquad \text{(dichotomy)}$$

- a *well-order* if, moreover, every non-empty subset of $X$ has a *least* element.

Let us write '$x < y$' for '$x \leq y \wedge x \neq y$'. A linear order $\leq$ on $X$ is

- a *dense order* if $X$ is not a singleton and for every $x, y \in X$

$$x < y \ \rightarrow\ \exists z\, (x < z \wedge z < y) \qquad \text{(density)}$$

# Functions

A relation $f$ is a function if every $x \in \mathrm{dom}(f)$ has exactly one $y$ with $(x,y) \in f$.

- We say that $y$ is the *value* of the function $f$ at $x$, denoted by $f(x) = y$,
- $f\colon X \to Y$ denotes that $f$ is a function with $\mathrm{dom}(f) = X$ and $\mathrm{rng}(f) \subseteq Y$,
- a function $f$ is a *surjection* (*onto* $Y$) if $\mathrm{rng}(f) = Y$,
- a function $f$ is *injection* (*one-to-one*) if for every $x, y \in \mathrm{dom}(f)$

$$x \neq y \quad \to \quad f(x) \neq f(y)$$

- $f\colon X \to Y$ is *bijection* from $X$ to $Y$ if it is both injection and surjection,
- if $f\colon X \to Y$ is injective, then $f^{-1} = \{(y,x) \mid (x,y) \in f\}$ is its *inverse*,
- the *image* of the set $A$ under $f$ is $f[A] = \{y \mid (x,y) \in f \text{ for some } x \in A\}$,
- if $f\colon X \to Y$ and $g\colon Y \to Z$, their composition $(f \circ g)\colon X \to Z$ satisfies

$$(f \circ g)(x) = g(f(x))$$

- ${}^{X}Y$ denotes the set of all functions from $X$ to $Y$.

# Numbers

We give examples of standard formal constructions.

- The natural numbers are defined inductively by $n = \{0, \ldots, n-1\}$, thus
$$0 = \emptyset, \quad 1 = \{0\} = \{\emptyset\}, \quad 2 = \{0, 1\} = \{\emptyset, \{\emptyset\}\}, \quad \ldots$$
- the sef of *natural* numbers $\mathbb{N}$ is defined as the smallest set containing $\emptyset$ which is closed under $S(x) := x \cup \{x\}$ (successor),
- the set of *integers* is $\mathbb{Z} = (\mathbb{N} \times \mathbb{N})/ \sim$, where $\sim$ is the equivalence
$$(a, b) \sim (c, d) \text{ if and only if } a + d = b + c$$
- the set of *rational* numbers is $\mathbb{Q} = (\mathbb{Z} \times (\mathbb{Z} \setminus \{0\}))/ \approx$, where $\approx$ is given by
$$(a, b) \approx (c, d) \text{ if and only if } a.d = b.c$$
- the set of *real* numbers $\mathbb{R}$ is the set of *cuts* of rational numbers, that is non-trivial downwards closed subsets of $\mathbb{Q}$ with no greatest element. ($A \subset \mathbb{Q}$ is *downwards closed* if $y < x \in A$ implies $y \in A$.)

# Cardinalities

- $x$ has *cardinality smaller or equal* to the cardinality of $y$ if there is an injective function $f: x \to y$, $(x \preccurlyeq y)$
- $x$ has *same cardinality* as $y$ if there is a bijection $f: x \to y$, $(x \approx y)$
- $x$ has *cardinality strictly smaller* than $y$ if $x \preccurlyeq y$ but not $x \approx y$, $(x \prec y)$

**Theorem (Cantor)** $x \prec \mathcal{P}(x)$ *for every set* $x$.

*Proof* $f(y) = \{y\}$ for $y \in x$ is an injective function $f: x \to \mathcal{P}(x)$, so $x \preccurlyeq \mathcal{P}(x)$.
Suppose for a contradiction that there is an injective $g: \mathcal{P}(x) \to x$. Define

$$y = \{g(z) \mid z \subseteq x \wedge g(z) \notin z\}$$

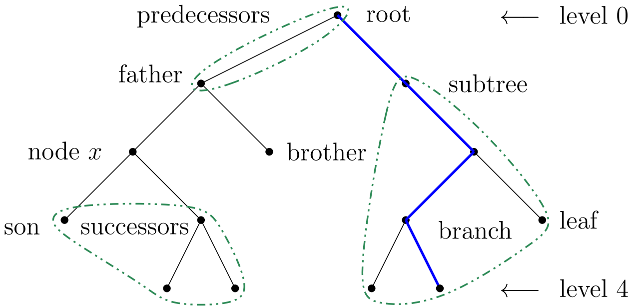
By definition, $g(y) \in y$ if and only if $g(y) \notin y$, a contradiction. $\square$

- for every $x$ there is *cardinal number* $\kappa$ with $x \approx \kappa$, denoted by $|x| = \kappa$,
- $x$ is *finite* if $|x| = n$ for some $n \in \mathbb{N}$; otherwise, $x$ is *infinite*,
- $x$ is *countable* if $x$ is finite or $|x| = |\mathbb{N}| = \omega$; otherwise, $x$ is *uncountable*,
- $x$ has *cardinality of the continuum* if $|x| = |\mathcal{P}(\mathbb{N})| = \mathfrak{c}$.

# $n$-ary relations and functions

- A relation of *arity* $n \in \mathbb{N}$ on $X$ is any set $R \subseteq X^n$, so for $n = 0$ we have either $R = \emptyset = 0$ or $R = \{\emptyset\} = 1$, and for $n = 1$ we have $R \subseteq X$,
- A (partial) function of *arity* $n \in \mathbb{N}$ from $X$ to $Y$ is any function $f \subseteq X^n \times Y$. We say that $f$ is *total* on $X^n$ if $\mathrm{dom}(f) = X^n$, denoted by $f \colon X^n \to Y$. If, moreover, $Y = X$, we say that $f$ is an *operation* on $X$.
- A function $f \colon X^n \to Y$ is *constant* if $\mathrm{rng}(f) = \{y\}$ for some $y \in Y$, for $n = 0$ we have $f = \{(\emptyset, y)\}$ and we identify $f$ with the *constant* $y$.
- The arity of a relation or function is denoted by $\mathrm{ar}(R)$ or $\mathrm{ar}(f)$ and we speak about *nullary*, *unary*, *binary*, etc. relations and functions.

# Trees




- A *tree* is a set $T$ with a partial order $<_T$ in which there is a unique least element, called the *root*, and the set of predecessors of any element is well ordered by $<_T$,
- a *branch* of a tree $T$ is a maximal linearly ordered subset of $T$,
- we adopt standard terminology on trees from the graph theory, e.g.

*a branch in a finite tree is a path from the root to a leaf.*

# König's lemma

We will consider *(for simplicity)* usually finitely branching trees in which every node except the root has an immediate predecessor (*father*).

- *$n$-th level* of a tree $T$ for $n \in \mathbb{N}$ is given by induction, it is the set of sons of nodes from the $(n-1)$-th level, $0$-th level containing exactly the root,
- the *depth* of $T$ is the maximal $n \in \mathbb{N}$ of non-empty level; if $T$ has infinite branch, then it has *infinite depth* $\omega$.
- a tree $T$ is *$n$-ary* for $n \in \mathbb{N}$ if every node has at most $n$ sons. It is *finitely branching*, if every node has only finitely many sons.

**Lemma (König)** *Every infinite, finitely branching tree contains an infinite branch.*

*Proof* We start in the root. Since it has only finitely many sons, there exists a son with infinitely many successors. We *choose* him and continue in his subtree. In this way we construct an infinite branch. □

# Ordered trees

- An *ordered tree* is a tree $T$ with a linear order of sons at each node. These orders are called *left-right orders* and are denoted by $<_L$. In comparison with $<_L$, the order $<_T$ is called the *tree order*.
- A *labeled tree* is a tree $T$ with an arbitrary function (a *labeling function*), that assigns to each node some object (a *label*).
- Labeled ordered trees represent, for example, structure of formulas.