Propositional and Predicate Logic - Tutorial 1

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1. Consider a finite game of two alternating players. Assume the game ends after n rounds by win of one of the players, denoted by X, Y where X begins. The game is given by formula $\varphi(x_1, y_1, x_2, y_2, \ldots, x_n, y_n)$ expressing that the game with moves $x_1, y_1, x_2, y_2, \ldots, x_n, y_n$ ends by a win of X (there are no draws). Find formulas (with use of quantifiers) that expresses
   (a) "X cannot lose", "Y cannot lose",
   (b) "X has a winning strategy", "Y has a winning strategy".
2. Assume we are given an (undirected) graph G and two his vertices u, v. Find a propositional formula that is satisfiable if and only if
   (a) G is bipartite,
   (b) G is 3-colorable,
   (c) G has a perfect matching,
   (d) there is a path between u and v in G.
3. Find first-order formulas (in the language of graph theory) expressing about an (undirected) graph that
   (a) "u and v have (a / exactly one / at most one) common neighbor",
   (b) "there are at least three mutually independent edges",
   (c) "there is a path of length n between u and v, where given $n > 0$ is fixed.
4. Find second-order formulas (in the language of graph theory) expressing about an (undirected) graph that
   (a) "there exists a bipartition",
   (b) "there exists a perfect matching",
   (c) "there exists a path between u and v".
5. Find first-order formulas (with the symbol $\leq$) expressing about a partially ordered set
   (a) "x is the smallest element", "x is a minimal element",
   (b) "x has an immediate successor",
   (c) "every two elements have the greatest common predecessor".
6. Find first order formulas (with use of equality) expressing for a fixed $n > 0$ that
   (a) "there exist at least n elements",
   (b) "there exist at most n elements",
   (c) "there exist exactly n elements"

   Is it possible to express with use of (possibly infinite) set of formulas that "there are infinitely many elements"?
7. Find a second-order formula expressing "there exist finitely many elements". Hint:
   (a) Find first-order formulas (with a symbol f for a function) expressing "f is injective", "f is surjective".
   (b) Find a second-order formula expressing "every function that is surjective is also injective".

8. Can we color the integers from 1 to n with two colors such that there is no monochromatic solution of an equation $a + b = c$ with $1 \leq a < b < c \leq n$? Write a proposition φ_n for $n = 8$ that is satisfiable if and only if such coloring exists.

9. *(Pigeonhole principle)*. Let $n \geq 2$ be a fixed natural number. Assume that we have n pigeons and $n - 1$ pigeonholes. We want to express that

   (i) Every pigeon sits in some pigeonhole,

   (ii) there is no pigeonhole with more than one pigeon sitting in it.

   Let $\mathbb{P} = \{p^i_j \mid 1 \leq i \leq n, 1 \leq j \leq n-1\}$ be a set of propositional variables, where p^i_j represents that *"the i-th pigeon sits in the j-th pigeonhole"*.

   (a) Write propositions φ_i and ψ_j over $\mathbb{P}$ expressing that *"the i-th pigeon sits in some pigeonhole"* and *"in the j-th pigeonhole sits not more than one pigeon"*, respectively, where $1 \leq i \leq n$, $1 \leq j \leq n-1$. Write a set of propositions expressing (i) and (ii).

10. Three proposals are being discussed in the parliament: *school charges*, *tax increase*, *restriction of smoking in restaurants.*

    (i) *The party A demands that in case the party B or the party C has his demand fulfilled, there will be no school charges or no tax increase.*

    (ii) *The party B wants to restrict smoking if the party C does not have his demand fulfilled or tax do not increase.*

    (iii) *The party C requires that in case the party A has his demand fulfilled, there will be no tax increase and no smoking restriction.*

    (iv) *In the final voting exactly two parties had their demands fulfilled.*

    Let the propositional letters p, q, r represent (respectively) that the proposals on *school charges, tax increase, smoking restriction* have been passed. Furthermore, let a, b, c represent (respectively) that each party's demand has been fulfilled and let $\mathbb{P} = \{p, q, r, a, b, c\}$.

    (a) Write propositions φ_1, φ_2, φ_3 in the form of an equivalence and a proposition φ_4 over $\mathbb{P}$ that express (respectively) (i), (ii), (iii), and (iv).

11. (In Shakespeare's play The Merchant of Venice) those who wish to win a beautiful Portia's hand in marriage need to find out which of the three caskets, made of gold, silver, and lead, hides Portia's portrait. We know that

    (i) Portia's portrait is in exactly one casket.

    (ii) At most one of the inscriptions on the caskets is true.

    (iii) The inscription on the golden casket says: *"The portrait is not in this casket."*

    (iv) The inscription on the silver casket says: *"If the inscription on the golden casket is true, then the portrait is in the leaden casket."*

    (v) The inscription on the leaden casket says: *"The inscription on the golden casket is false."*

    Let the propositional letters g, s, l represent (respectively) that *"the portrait is in golden / silver / leaden casket"* and letters t_g, t_s, t_l represent (respectively) that *"the inscription on golden / silver / leaden casket is true."* Furthermore, let $\mathbb{P}' = \{g, s, l, t_g, t_s, t_l\}$.

    (a) Write propositions φ_1, φ_2 over $\mathbb{P}'$ expressing the statements (i), (ii), and propositions (in form of equivalences) φ_3, φ_4, φ_5 over $\mathbb{P}'$ representing (respectively) our knowledge from (iii), (iv), (v).