Predicate and Propositional Logic - Tutorials 2,3

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1. Consider a theory $T = \{\neg q \to (\neg p \vee q),\ \neg p \to q,\ r \to q\}$. Which of the following propositions are valid, contradictory, independent, satisfiable, equivalent in T?

   (a) p, q, r, s

   (b) $p \vee q$, $p \vee r$, $p \vee s$, $q \vee s$

   (c) $p \wedge q$, $q \wedge s$, $p \to q$, $s \to q$

2. Prove or disprove that the following sets of connectives are universal.

   (a) $\{\downarrow\}$ where $\downarrow$ is Peirce arrow (NOR)

   (b) $\{\uparrow\}$ where $\uparrow$ is Sheffer stroke (NAND)

   (c) $\{\vee, \to, \leftrightarrow\}$, $\{\vee, \wedge, \to\}$

3. Transform the following propositions into DNF and CNF a) by using truth tables (determining the models), b) by using transformation rules.

   (a) $(\neg p \vee q) \to (\neg q \wedge r)$

   (b) $(\neg p \to (\neg q \to r)) \to p$

   (c) $((p \to \neg q) \to \neg r) \to \neg p$

4. Let $\mathrm{maj}_n \colon (\{0,1\}^n)^3 \to \{0,1\}^n$ be the coordinate-wise majority function; that is, for example

   $$\mathrm{maj}_4((0,1,0,1),(1,1,0,0),(1,1,0,0)) = (1,1,0,0)$$

   We say that a set $K \subseteq \{0,1\}^n$ is a *median* set if it is closed under maj_n.

   (a) Show that for every 2-CNF proposition φ it holds that $M(\varphi)$ is a median set.

   (b)* Show that for every median set $K \subseteq \{0,1\}^n$ there exists a 2-CNF proposition φ over n variables such that $M(\varphi) = K$.

5. Consider an infinite theory $T = \{p_i \to (p_{i+1} \vee q_{i+1}), q_i \to (p_{i+1} \vee q_{i+1}) \mid i \in \mathbb{N}\}$ over $\mathrm{var}(T)$.

   (a) Which propositions in the form $p_i \to p_j$ are logical consequences of T?

   (b) Which propositions in the form $p_i \to (p_j \vee q_j)$ are logical consequences of T?

   (c) Determine all models of the theory T.

6. Prove or disprove (or find the correct relation) that for every theory T and propositions φ, ψ over $\mathbb{P}$ it holds

   (a) $T \models \varphi$, if and only if $T \not\models \neg\varphi$

   (b) $T \models \varphi$ and $T \models \psi$, if and only if $T \models \varphi \wedge \psi$

   (c) $T \models \varphi$ or $T \models \psi$, if and only if $T \models \varphi \vee \psi$

   (d) $T \models \varphi \to \psi$ and $T \models \psi \to \chi$, if and only if $T \models \varphi \to \chi$

7. Prove or disprove (or find the correct relation). For every theories T and S over $\mathbb{P}$

   (a) $S \subseteq T \Rightarrow \theta^{\mathbb{P}}(T) \subseteq \theta^{\mathbb{P}}(S)$

   (b) $\theta^{\mathbb{P}}(S \cup T) = \theta^{\mathbb{P}}(S) \cup \theta^{\mathbb{P}}(T)$

   (c) $\theta^{\mathbb{P}}(S \cap T) = \theta^{\mathbb{P}}(S) \cap \theta^{\mathbb{P}}(T)$

8. Let $|\mathbb{P}| = n$ and $\varphi \in \mathrm{VF}_{\mathbb{P}}$ with $|M(\varphi)| = m$.

(a) What is the number of nonequivalent propositions ψ such that $\varphi \models \psi$ or $\psi \models \varphi$?

(b) What is the number of nonequivalent theories over $\mathbb{P}$ in which φ is valid? What is the number of nonequivalent *complete* theories over $\mathbb{P}$ in which φ is valid?

(c) What is the number of nonequivalent theories T over $\mathbb{P}$ such that $T \cup \{\varphi\}$ is satisfiable?

(d) Let, moreover, $\{\varphi, \psi\}$ be an unsatisfiable theory with $|M(\psi)| = p$. What is the number of nonequivalent propositions χ such that $\varphi \vee \psi \models \chi$? What is the number of nonequivalent theories in which $\varphi \vee \psi$ is valid?

9. Let $T = \{q \to (\neg p \to r),\ \neg r \to (\neg p \wedge q),\ (s \to r) \to p\}$ be a theory over the language $\mathbb{P} = \{p, q, r, s\}$.

   (a) Axiomatize the theory T by a proposition in CNF.

   (b) Find all models of the theory T.

   (c) Is the theory T an extension of the theory $S = \{q \leftrightarrow \neg r\}$ over the language $\{q, r\}$? Is T a conservative extension of S? Justify.

   (d) Determine the number of mutually inequivalent propositions in the language $\mathbb{P}$ that are *contradictory* in both the theories T and S. Justify.

10. Let $T = \{p \vee q \to r, \neg(p \to \neg s)\}$ be a theory over the propositional language $\mathbb{P} = \{p, q, r, s\}$.

    (a) Is the proposition $q \to p$ valid in the theory T? Is it contradictory? Is it independent? Justify.

    (b) Find all models of the theory T.

    (c) Find a theory S over the language $\mathbb{P}' = \{p, q, r\}$ such that T is a conservative extension of S. Axiomatize S by a proposition in CNF. Justify why S has the desired property.

    (d) Determine the number of mutually inequivalent propositions φ over $\mathbb{P}$ such that φ is valid in T and independent in S.

11. Let $T = \{p,\ \neg q \to \neg r,\ \neg q \to \neg s,\ r \to p,\ \neg s \to \neg p\}$ be a theory over the language $\mathbb{P} = \{p, q, r, s\}$.

    (a) Using the implication graph, show that T is satisfiable.

    (b) Find all models of the theory T and axiomatize $M^{\mathbb{P}}(T)$ by a proposition in CNF.

    (c) Determine, and justify, the number of mutually

       (i) inequivalent propositions over $\mathbb{P}$ that are independent in T,

       (ii) T-inequivalent propositions over $\mathbb{P}$ which are independent in T.

12. Let $T = \{(r \to p) \to \neg q,\ \neg q \to p, \neg(r \wedge q), r \to \neg s\}$ be a theory in the language $\mathbb{P} = \{p, q, r, s\}$.

    (a) Axiomatize $M^{\mathbb{P}}(T)$ by a proposition in CNF.

    (b) Is the theory T a conservative extension of some theory over the language $\{p, q, r\}$? Justify.

    (c) Determine the number of mutually inequivalent noncontradictory extensions of the theory T over the language $\{p, q, r, s, t\}$. Justify.

13. Let $T = \{p \to \neg q \wedge r, q \vee r, (q \wedge s) \leftrightarrow r\}$ be a theory over the language $\mathbb{P} = \{p, q, r, s\}$.

    (a) Is the proposition $q \to p$ valid in the theory T? Is it contradictory? Is it independent? Justify your answers.

    (b) Axiomatize $M(T)$ by a proposition in CNF.

    (c) Determine the number of mutually inequivalent theories S over $\mathbb{P}' = \{r, s\}$ such that T is a conservative extension of S. How many of them are complete?