Predicate and Propositional Logic - Tutorial 7

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1. Which of the variable occurrences are free/bound in the following formulas? Find variants of these formulas without variables that have both free and bound occurrence.

   (a) $(\exists x)(\forall y)P(y,z) \vee (y = 0)$

   (b) $(\exists x)(P(x) \wedge (\forall x)Q(x)) \vee (x = 0)$

   (c) $(\exists x)(x > y) \wedge (\exists y)(y > x)$

2. Let φ denote the formula $(\forall x)((x = z) \vee (\exists y)(f(x) = y) \vee (\forall z)(y = f(z)))$. Which of the following terms are substitutable into φ?

   (a) the term z for the variable x, the term y for the variable x,

   (b) the term z for the variable y, the term $2 * y$ for the variable y,

   (c) the term x for the variable z, the term y for the variable z,

3. Are the following formulas variants of the formula $(\forall x)(x < y \vee (\exists z)(z = y \wedge z \neq x))$?

   (a) $(\forall z)(z < y \vee (\exists z)(z = y \wedge z \neq z))$

   (b) $(\forall y)(y < y \vee (\exists z)(z = y \wedge z \neq y))$

   (c) $(\forall u)(u < y \vee (\exists z)(z = y \wedge z \neq u))$

4. Let $\mathcal{A} = (\{a,b,c,d\}, \rhd^A)$ be a structure for the language with a single binary relation symbol $\rhd$, where $\rhd^A = \{(a,c),(b,c),(c,c),(c,d)\}$. Which of the following formulas are valid in $\mathcal{A}$?

   (a) $x \rhd y$

   (b) $(\exists x)(\forall y)(y \rhd x)$

   (c) $(\exists x)(\forall y)((y \rhd x) \to (x \rhd x))$

   (d) $(\forall x)(\forall y)(\exists z)((x \rhd z) \wedge (z \rhd y))$

   (e) $(\forall x)(\exists y)((x \rhd z) \vee (z \rhd y))$

5. For every formula φ from the previous exercise find a structure $\mathcal{B}$ (if it exists) such that $\mathcal{B} \models \varphi$ if and only if $\mathcal{A} \not\models \varphi$.

6. Are the following sentences valid / contradictory / independent (in logic)?

   (a) $(\exists x)(\forall y)(P(x) \vee \neg P(y))$

   (b) $(\forall x)(P(x) \to Q(f(x))) \wedge (\forall x)P(x) \wedge (\exists x)\neg Q(x)$

   (c) $(\forall x)(P(x) \vee Q(x)) \to ((\forall x)(P(x) \vee (\forall x)Q(x))$

   (d) $(\forall x)(P(x) \to Q(x)) \to ((\exists x)P(x) \to (\exists x)Q(x))$

   (e) $(\exists x)(\forall y)P(x,y) \to (\forall y)(\exists x)P(x,y)$

7. Prove (semantically) the following claims. For every structure $\mathcal{A}$, formula φ, and sentence ψ,

   (a) $\mathcal{A} \models (\psi \to (\exists x)\varphi) \Leftrightarrow \mathcal{A} \models (\exists x)(\psi \to \varphi)$

   (b) $\mathcal{A} \models (\psi \to (\forall x)\varphi) \Leftrightarrow \mathcal{A} \models (\forall x)(\psi \to \varphi)$

   (c) $\mathcal{A} \models ((\exists x)\varphi \to \psi) \Leftrightarrow \mathcal{A} \models (\forall x)(\varphi \to \psi)$

   (d) $\mathcal{A} \models ((\forall x)\varphi \to \psi) \Leftrightarrow \mathcal{A} \models (\exists x)(\varphi \to \psi)$

   Does this hold also for every formula ψ with a free variable x? And for every formula ψ in which x is not free?