Predicate and Propositional Logic - Seminar 8

Nov 21, 2024

1. Let $\mathcal{A} = (\{a, b, c, d\}, \rhd^A)$ be a structure for the language with a single binary relation symbol $\rhd$, where $\rhd^A = \{(a, c), (b, c), (c, c), (c, d)\}$. Which of the following formulas are valid in $\mathcal{A}$?
   (a) $x \rhd y$
   (b) $(\exists x)(\forall y)(y \rhd x)$
   (c) $(\exists x)(\forall y)((y \rhd x) \to (x \rhd x))$
   (d) $(\forall x)(\forall y)(\exists z)((x \rhd z) \wedge (z \rhd y))$
   (e) $(\forall x)(\exists y)((x \rhd z) \vee (z \rhd y))$
2. For every formula φ from the previous exercise find a structure $\mathcal{B}$ (if it exists) such that $\mathcal{B} \models \varphi$ if and only if $\mathcal{A} \not\models \varphi$.
3. Are the following sentences valid / contradictory / independent (in logic)?
   (a) $(\exists x)(\forall y)(P(x) \vee \neg P(y))$
   (b) $(\forall x)(P(x) \to Q(f(x))) \wedge (\forall x)P(x) \wedge (\exists x)\neg Q(x)$
   (c) $(\forall x)(P(x) \vee Q(x)) \to ((\forall x)(P(x) \vee (\forall x)Q(x))$
   (d) $(\forall x)(P(x) \to Q(x)) \to ((\exists x)P(x) \to (\exists x)Q(x))$
   (e) $(\exists x)(\forall y)P(x, y) \to (\forall y)(\exists x)P(x, y)$
4. Prove (semantically) the following claims. For every structure $\mathcal{A}$, formula φ, and sentence ψ,
   (a) $\mathcal{A} \models (\psi \to (\exists x)\varphi) \Leftrightarrow \mathcal{A} \models (\exists x)(\psi \to \varphi)$
   (b) $\mathcal{A} \models (\psi \to (\forall x)\varphi) \Leftrightarrow \mathcal{A} \models (\forall x)(\psi \to \varphi)$
   (c) $\mathcal{A} \models ((\exists x)\varphi \to \psi) \Leftrightarrow \mathcal{A} \models (\forall x)(\varphi \to \psi)$
   (d) $\mathcal{A} \models ((\forall x)\varphi \to \psi) \Leftrightarrow \mathcal{A} \models (\exists x)(\varphi \to \psi)$

   Does this hold also for every formula ψ with a free variable x? And for every formula ψ in which x is not free?
5. Determine whether the following holds for every formula φ.
   (a) $\varphi \models (\forall x)\varphi$
   (b) $\models \varphi \to (\forall x)\varphi$
   (c) $\varphi \models (\exists x)\varphi$
   (d) $\models \varphi \to (\exists x)\varphi$
6. The theory of groups T is of language $L = \langle +, -, 0\rangle$ with equality where $+$ is a binary function symbol, $-$ is a unary function symbol, 0 is a constant symbol, and has it axioms

$$x + (y + z) = (x + y) + z$$
$$0 + x = x = x + 0$$
$$x + (-x) = 0 = (-x) + x$$

   Are the following formulas valid / contradictory / independent in T?
   (a) $x + y = y + x$
   (b) $x + y = x \ \to \ y = 0$
   (c) $x + y = 0 \ \to \ y = -x$

(d) $-(x+y) = (-y) + (-x)$

7. Consider a structure $\underline{\mathbb{Z}_4} = \langle \{0,1,2,3\}, +, -, 0 \rangle$ where $+$ is the binary addition modulo 4 and $-$ is the unary function for the *inverse* element of $+$ with respect to the *neutral* element 0.

   (a) Is $\underline{\mathbb{Z}_4}$ a model of the theory T from the previous example (i.e. is it a *group*)?

   (b) Determine all substructures $\underline{\mathbb{Z}_4}\langle a \rangle$ generated by some $a \in \mathbb{Z}_4$.

   (c) Does $\underline{\mathbb{Z}_4}$ contain also other substructures?

   (d) Is every substructure of $\underline{\mathbb{Z}_4}$ a model of T?

   (e) Is every substructure of $\underline{\mathbb{Z}_4}$ elementarily equivalent to $\underline{\mathbb{Z}_4}$?

   (f) Is every substructure of a *commutative* group (i.e. a group that satisfies also 2(a)) again a commutative group?

8. Let $\underline{\mathbb{Q}} = \langle \mathbb{Q}, +, -, \cdot, 0, 1 \rangle$ be the structure of rational numbers with standard operations (thus forming a *field*).

   (a) Is there a reduct of $\underline{\mathbb{Q}}$ that is a model of T from the previous exercises?

   (b) Can we expand the reduct $\langle \mathbb{Q}, \cdot, 1 \rangle$ to a model of T?

   (c) Does $\underline{\mathbb{Q}}$ contain a substructure that is not elementary equivalent to $\underline{\mathbb{Q}}$?

   (d) Let $Th(\underline{\mathbb{Q}})$ denote the set of all sentences that are valid in $\underline{\mathbb{Q}}$. Is $Th(\underline{\mathbb{Q}})$ a complete theory?

9. Let $T = \{x = c_1 \vee x = c_2 \vee x = c_3\}$ be a theory of $L = \langle c_1, c_2, c_3 \rangle$ with equality.

   (a) Is T (semantically) consistent?

   (b) Are all models of T elementarily equivalent? That is, is T (semantically) complete?

   (c) Find all simple complete extensions of T.

   (d) Is a theory $T' = T \cup \{x = c_1 \vee x = c_4\}$ of the language $L = \langle c_1, c_2, c_3, c_4 \rangle$ an extension of T? Is T' a simple extension of T? Is T' a conservative extension of T?