Predicate and Propositional Logic - Tutorials 11,12

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1. Let φ, ψ be sentences or formulas in a free variable x, denoted by $\varphi(x)$, $\psi(x)$. Find tableau proofs of the following formulas.
   (a) $(\exists x)(\varphi(x) \lor \psi(x)) \leftrightarrow (\exists x)\varphi(x) \lor (\exists x)\psi(x)$,
   (b) $(\forall x)(\varphi(x) \land \psi(x)) \leftrightarrow (\forall x)\varphi(x) \land (\forall x)\psi(x)$,
   (c) $(\varphi \lor (\forall x)\psi(x)) \to (\forall x)(\varphi \lor \psi(x))$ where x is not free in φ,
   (d) $(\varphi \land (\exists x)\psi(x)) \to (\exists x)(\varphi \land \psi(x))$ where x is not free in φ.
   (e) $(\exists x)(\varphi \to \psi(x)) \to (\varphi \to (\exists x)\psi(x))$ where x is not free in φ,
   (f) $(\exists x)(\varphi \land \psi(x)) \to (\varphi \land (\exists x)\psi(x))$ where x is not free in φ,
   (g) $(\exists x)(\varphi(x) \to \psi) \to ((\forall x)\varphi(x) \to \psi)$ where x is not free in ψ,
   (h) $((\exists x)\varphi(x) \to \psi) \to (\forall x)(\varphi(x) \to \psi)$ where x is not free in ψ.

2. Let T^* be a theory with axioms of equality. Prove by tableau method that
   (a) $T^* \models x = y \ \to\ y = x$ (symmetry of $=$)
   (b) $T^* \models (x = y \ \land\ y = z) \ \to\ x = z$ (transitivity of $=$)

   Hint: To show (a) apply the axiom of equality (iii) for $x_1 = x$, $x_2 = x$, $y_1 = y$ a $y_2 = x$, to show (b) apply (iii) for $x_1 = x$, $x_2 = y$, $y_1 = x$ a $y_2 = z$.

3. Convert the following formulas into the prenex normal form.
   (a) $(\forall y)((\exists x)P(x, y) \to Q(y, z)) \land (\exists y)((\forall x)R(x, y) \lor Q(x, y))$
   (b) $(\exists x)R(x, y) \leftrightarrow (\forall y)P(x, y)$
   (c) $\neg((\forall x)(\exists y)P(x, y) \to (\exists x)(\exists y)R(x, y)) \land (\forall x)\neg(\exists y)Q(x, y)$

4. Find Skolem variants of the formulas in PNF from the previous problem.

5. Verify that (thus, a Skolem variant does not have to be equivalent to the original formula)
   (a) $\models (\forall x)P(x, f(x)) \to (\forall x)(\exists y)P(x, y)$
   (b) $\not\models (\forall x)(\exists y)P(x, y) \to (\forall x)P(x, f(x))$

6. The theory T of *fields* in $L = \langle +, -, \cdot, 0, 1\rangle$ contains one axiom φ that is not open:

   $$x \neq 0 \ \to\ (\exists y)(x \cdot y = 1).$$

   We know that $T \models 0 \cdot y = 0$ and $T \models\ (x \neq 0 \ \land\ x \cdot y = 1 \ \land\ x \cdot z = 1) \ \to\ y = z$.
   (a) Find a Skolem variant φ_S of φ with a new function symbol f.
   (b) Let T' be the theory obtained from T by replacing φ with φ_S. Is $T' \models \varphi$?
   (c) Can every model of T be *uniquely* expanded to a model of T'?

7. Let T denote the (previous) theory of fields. Let ψ be the formula $x \cdot y = 1 \lor (x = 0 \land y = 0)$.
   (a) Do the conditions of existence and uniqueness hold in T for $\psi(x, y)$ and the variable y?
   (b) Find an extension T^* of T by definition of a function symbol f with the formula ψ.
   (c) Is T^* equivalent to the theory T' from the previous problem?
   (d) Find a formula of the original language L that is equivalent in T^* to the formula

   $$f(x \cdot y) = f(x) \cdot f(y)$$

8. Find Herbrand universe and an example of a Herbrand structure for the following languages.
   (a) $L = \langle P, Q, f, a, b \rangle$ where P, Q are unary resp. binary relation symbols, f is a unary function symbol, a, b are constant symbols.
   (b) $L = \langle P, f, g, a \rangle$ where P is a binary relation s., f, g unary function s., a constant symbol.
9. Find Herbrand models for the following theories or find unsatisfiable conjunctions of ground instances of their axioms. Assume that the language has constant symbols a, b.
   (a) $T = \{\neg P(x) \vee Q(f(x), y), \neg Q(x, b), P(a)\}$
   (b) $T = \{\neg P(x) \vee Q(f(x), y), Q(x, b), P(a)\}$
   (c) $T = \{P(x, f(x)), \neg P(x, g(x))\}$
   (d) $T = \{P(x, f(x)), \neg P(x, g(x)), P(g(x), f(y)) \to P(x, y)\}$
10. Transform the following formulas to equisatisfiable formulas in clausal form.
    (a) $(\forall y)(\exists x)P(x, y)$
    (b) $\neg(\forall y)(\exists x)P(x, y)$
    (c) $\neg(\exists x)((P(x) \to P(a)) \wedge (P(x) \to P(b)))$
    (d) $(\exists x)(\forall y)(\exists z)(P(x, z) \wedge P(z, y) \to R(x, y))$
11. Find (all) resolvents of the following pairs of clauses.
    (a) $\{P(x, y), P(y, z)\}$, $\{\neg P(u, f(u))\}$
    (b) $\{P(x, x), \neg R(x, f(x))\}$, $\{R(x, y), Q(y, z)\}$
    (c) $\{P(x, y), \neg P(x, x), Q(x, f(x), z)\}$, $\{\neg Q(f(x), x, z), P(x, z)\}$
12. Show that the following set of clauses if resolution refutable. Describe the resolution refutation by a resolution tree. In each resolution step write down the unification used and underline resolved literals.
    (a) $\{P(a, x, f(y)), P(a, z, f(h(b))), \neg Q(y, z)\}$
    (b) $\{\neg Q(h(b), w), H(w, a)\}$
    (c) $\{\neg P(a, w, f(h(b))), H(x, a)\}$
    (d) $\{P(a, u, f(h(u))), H(u, a), Q(h(b), b)\}$
    (e) $\{\neg H(v, a)\}$
13. We know that
    (a) If a brick is on (another) brick, then it is not on the ground.
    (b) Every brick is on (another) brick or on the ground.
    (c) No brick is on a brick that is on (another) brick.

    Express these facts in a first-order language and prove by resolution that if a brick is on another brick, the lower brick is on the ground.
14. We know that
    (a) Every barber shaves all who do not shave themselves.
    (b) No barber shaves someone who shaves himself.

    Express these facts in a first-order language and prove by resolution that no barber exists.