Propositional and Predicate Logic - VIII

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Validity in a substructure

Let $\mathcal{B}$ be a substructure of a structure $\mathcal{A}$ for a (fixed) language L.

Proposition *For every open formula φ and assignment $e\colon \mathrm{Var} \to B$,*

$$\mathcal{A} \models \varphi[e] \quad \textit{if and only if} \quad \mathcal{B} \models \varphi[e].$$

Proof For atomic φ it follows from the definition of the truth value with respect to an assignment. Otherwise by induction on the structure of the formula. □

Corollary *For every open formula φ and structure $\mathcal{A}$,*

$$\mathcal{A} \models \varphi \quad \textit{if and only if} \quad \mathcal{B} \models \varphi \textit{ for every substructure } \mathcal{B} \subseteq \mathcal{A}.$$

- A theory T is *open* if all axioms of T are open.

Corollary *Every substructure of a model of an open theory T is a model of T.*

For example, every substructure of a graph, i.e. a model of theory of graphs, is a graph, called a subgraph. Similarly subgroups, Boolean subalgebras, etc.

Theorem on constants

Theorem *Let φ be a formula in a language L with free variables $x_1, \ldots, x_n$ and let T be a theory in L. Let L' be the extension of L with new constant symbols $c_1, \ldots, c_n$ and let T' denote the theory T in L'. Then*

$$T \models \varphi \quad \textit{if and only if} \quad T' \models \varphi(x_1/c_1, \ldots, x_n/c_n).$$

Proof ($\Rightarrow$) If $\mathcal{A}'$ is a model of T', let $\mathcal{A}$ be the reduct of $\mathcal{A}'$ to L. Since $\mathcal{A} \models \varphi[e]$ for every assignment e, we have in particular

$$\mathcal{A} \models \varphi[e(x_1/c_1^{A'}, \ldots, x_n/c_n^{A'})], \quad \text{i.e. } \mathcal{A}' \models \varphi(x_1/c_1, \ldots, x_n/c_n).$$

($\Leftarrow$) If $\mathcal{A}$ is a model of T and e an assignment, let $\mathcal{A}'$ be the expansion of A into L' by setting $c_i^{A'} = e(x_i)$ for every i. Since $\mathcal{A}' \models \varphi(x_1/c_1, \ldots, x_n/c_n)[e']$ for every assignment e', we have

$$\mathcal{A}' \models \varphi[e(x_1/c_1^{A'}, \ldots, x_n/c_n^{A'})], \quad \text{i.e. } \mathcal{A} \models \varphi[e]. \quad \square$$

Boolean algebras

The theory of *Boolean algebras* has the language $L = \langle -, \wedge, \vee, 0, 1\rangle$ with equality and the following axioms.

$$x \wedge (y \wedge z) = (x \wedge y) \wedge z \qquad \text{(asociativity of } \wedge)$$
$$x \vee (y \vee z) = (x \vee y) \vee z \qquad \text{(asociativity of } \vee)$$
$$x \wedge y = y \wedge x \qquad \text{(commutativity of } \wedge)$$
$$x \vee y = y \vee x \qquad \text{(commutativity of } \vee)$$
$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \qquad \text{(distributivity of } \wedge \text{ over } \vee)$$
$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \qquad \text{(distributivity of } \vee \text{ over } \wedge)$$
$$x \wedge (x \vee y) = x, \quad x \vee (x \wedge y) = x \qquad \text{(absorption)}$$
$$x \vee (-x) = 1, \quad x \wedge (-x) = 0 \qquad \text{(complementation)}$$
$$0 \neq 1 \qquad \text{(non-triviality)}$$

The smallest model is $\underline{2} = \langle 2, -_1, \wedge_1, \vee_1, 0, 1\rangle$. Finite Boolean algebras are (up to isomorphism) exactly $\underline{^n2} = \langle {^n2}, -_n, \wedge_n, \vee_n, 0_n, 1_n\rangle$ for $n \in \mathbb{N}^+$, where the operations *(on binary n-tuples)* are the coordinate-wise operations of $\underline{2}$.

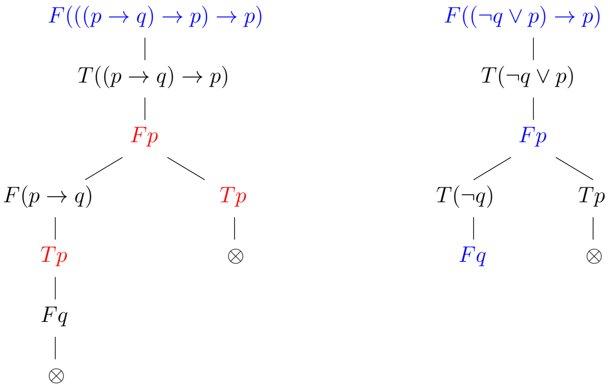
Relations of propositional and predicate logic

- Propositional formulas over connectives $\neg$, $\wedge$, $\vee$ (eventually with $\top$, $\bot$) can be viewed as Boolean terms. Then the truth value of φ in a given assignment is the value of the term in the Boolean algebra $\underline{2}$.
- Lindenbaum-Tarski algebra over $\mathbb{P}$ is Boolean algebra (also for $\mathbb{P}$ infinite).
- If we represent atomic subformulas in an open formula φ (without equality) with propositional letters, we obtain a proposition that is valid if and only if φ is valid.
- Propositional logic can be introduced as a fragment of predicate logic using nullary relation symbols (*syntax*) and nullary relations (*semantics*) since $A^0 = \{\emptyset\} = 1$, so $R^A \subseteq A^0$ is either $R^A = \emptyset = 0$ or $R^A = \{\emptyset\} = 1$.

Tableau method in propositional logic - a review

- A tableau is a binary tree that represents a search for a *counterexample*.
- Nodes are labeled by entries, i.e. formulas with a sign T / F that represents an assumption that the formula is true / false in some model.
- If this assumption is correct, then it is correct also for all the entries in some branch below that came from this entry.
- A branch is contradictory (it fails) if it contains $T\psi$, $F\psi$ for some ψ.
- A proof of formula φ is a contradictory tableau with root $F\varphi$, i.e. a tableau in which every branch is contradictory. If φ has a proof, it is valid.
- If a counterexample exists, there will be a branch in a finished tableau that provides us with this counterexample, but this branch can be infinite.
- We can construct a systematic tableau that is always finished.
- If φ is valid, the systematic tableau for φ is contradictory, i.e. it is a proof of φ; and in this case, it is also finite.

Tableau method in propositional logic - examples



$a)$ A tableau proof of the formula $((p \to q) \to p) \to p$.

$b)$ A finished tableau for $(\neg q \vee p) \to p$. The left branch provides us with a counterexample $v(p) = v(q) = 0$.

Tableau method in predicate logic - what is different

- Formulas in entries will always be sentences (closed formulas), i.e. formulas without free variables.
- We add new atomic tableaux for quantifiers.
- In these tableaux we substitute ground terms for quantified variables following certain rules.
- We extend the language by new (auxiliary) constant symbols (countably many) to represent *"witnesses"* of entries $T(\exists x)\varphi(x)$ and $F(\forall x)\varphi(x)$.
- In a finished branch containing an entry $T(\forall x)\varphi(x)$ or $F(\exists x)\varphi(x)$ we have instances $T\varphi(x/t)$ resp. $F\varphi(x/t)$ for every ground term t (of the extended language).

Assumptions

1) *The formula φ that we want to prove (or refute) is a sentence.* If not, we can replace φ with its universal closure φ', since for every theory T,

$$T \models \varphi \quad \text{if and only if} \quad T \models \varphi'.$$

2) *We prove from a theory in a closed form, i.e. every axiom is a sentence.* By replacing every axiom ψ with its universal closure ψ' we obtain an equivalent theory since for every structure $\mathcal{A}$ (of the given language L),

$$\mathcal{A} \models \psi \quad \text{if and only if} \quad \mathcal{A} \models \psi'.$$

3) *The language L is at most countable.* Then every theory of L is at most countable. We denote by L_C the extension of L by new constant symbols $c_0, c_1, \ldots$ (countably many). Then there are countable many ground terms of L_C. Let t_i denote the i-th ground term (in some fixed enumeration).

4) *First, we assume that the language is without equality.*

Tableaux in predicated logic - examples

$F((\exists x)\neg P(x) \to \neg(\forall x)P(x))$

$T(\exists x)\neg P(x)$

$F(\neg(\forall x)P(x))$

$T(\forall x)P(x)$

$T(\neg P(c))$ c new

$FP(c)$

$T(\forall x)P(x)$

$TP(c)$

$\otimes$

$F(\neg(\forall x)P(x) \to (\exists x)\neg P(x))$

$T(\neg(\forall x)P(x))$

$F(\exists x)\neg P(x)$

$F(\forall x)P(x)$

$FP(d)$ d new

$F(\exists x)\neg P(x)$

$F(\neg P(d))$

$TP(d)$

$\otimes$

Atomic tableaux - original

An *atomic tableau* is one of the following trees (labeled by entries), where α is any atomic sentence and φ, ψ are any sentences, all of language L_C.

$T\alpha$	$F\alpha$	$\begin{array}{c} T(\varphi\land\psi) \\ \mid \\ T\varphi \\ \mid \\ T\psi \end{array}$	$\begin{array}{ccc} & F(\varphi\land\psi) & \\ / & & \backslash \\ F\varphi & & F\psi \end{array}$	$\begin{array}{ccc} & T(\varphi\lor\psi) & \\ / & & \backslash \\ T\varphi & & T\psi \end{array}$	$\begin{array}{c} F(\varphi\lor\psi) \\ \mid \\ F\varphi \\ \mid \\ F\psi \end{array}$
$\begin{array}{c} T(\neg\varphi) \\ \mid \\ F\varphi \end{array}$	$\begin{array}{c} F(\neg\varphi) \\ \mid \\ T\varphi \end{array}$	$\begin{array}{ccc} & T(\varphi\to\psi) & \\ / & & \backslash \\ F\varphi & & T\psi \end{array}$	$\begin{array}{c} F(\varphi\to\psi) \\ \mid \\ T\varphi \\ \mid \\ F\psi \end{array}$	$\begin{array}{ccc} & T(\varphi\leftrightarrow\psi) & \\ / & & \backslash \\ T\varphi & & F\varphi \\ \mid & & \mid \\ T\psi & & F\psi \end{array}$	$\begin{array}{ccc} & F(\varphi\leftrightarrow\psi) & \\ / & & \backslash \\ T\varphi & & F\varphi \\ \mid & & \mid \\ F\psi & & T\psi \end{array}$

Atomic tableaux - new

Atomic tableaux are also the following trees (labeled by entries), where φ is any formula of the language L_C with a free variable x, t is any ground term of L_C and c is a new constant symbol from $L_C \setminus L$.

$\sharp$ $T(\forall x)\varphi(x)$ $\vert$ $T\varphi(x/t)$ for any ground term t of L_C	$*$ $F(\forall x)\varphi(x)$ $\vert$ $F\varphi(x/c)$ for a *new* constant c	$*$ $T(\exists x)\varphi(x)$ $\vert$ $T\varphi(x/c)$ for a *new* constant c	$\sharp$ $F(\exists x)\varphi(x)$ $\vert$ $F\varphi(x/t)$ for any ground term t of L_C

Remark *The constant symbol c represents a "witness" of the entry $T(\exists x)\varphi(x)$ or $F(\forall x)\varphi(x)$. Since we need that no prior demands are put on c, we specify (in the definition of a tableau) which constant symbols c may be used.*

Tableau

A *finite tableau* from a theory T is a binary tree labeled with entries described

- (i) every atomic tableau is a finite tableau from T, whereas in case $(*)$ we may use any constant symbol $c \in L_C \setminus L$,
- (ii) if P is an entry on a branch V in a finite tableau from T, then by adjoining the atomic tableau for P at the end of branch V we obtain (again) a finite tableau from T, whereas in case $(*)$ we may use only a constant symbol $c \in L_C \setminus L$ that does not appear on V,
- (iii) if V is a branch in a finite tableau from T and $\varphi \in T$, then by adjoining $T\varphi$ at the end of branch V we obtain (again) a finite tableau from T.
- (iv) every finite tableau from T is formed by finitely many steps (i), (ii), (iii).

A *tableau* from T is a sequence $\tau_0, \tau_1, \dots, \tau_n, \dots$ of finite tableaux from T such that τ_{n+1} is formed from τ_n by (ii) or (iii), formally $\tau = \cup \tau_n$.

Construction of tableaux

$$
\begin{array}{c}
\boxed{\begin{array}{c} F((\exists x)\neg P(x) \to \neg(\forall x)P(x)) \\ | \\ T(\exists x)\neg P(x) \\ | \\ F(\neg(\forall x)P(x)) \end{array}} \\
| \\
\boxed{\begin{array}{c} F(\neg(\forall x)P(x)) \\ | \\ T(\forall x)P(x) \end{array}} \\
| \\
\boxed{\begin{array}{c} T(\exists x)\neg P(x) \\ | \\ T(\neg P(c)) \quad c \text{ new} \end{array}} \\
| \\
\boxed{\begin{array}{c} T(\neg P(c)) \\ | \\ FP(c) \end{array}} \\
| \\
\boxed{\begin{array}{c} T(\forall x)P(x) \\ | \quad \text{choose} \\ TP(c) \quad t = c \end{array}} \\
| \\
\otimes
\end{array}
\qquad\qquad
\begin{array}{c}
\boxed{\begin{array}{c} F(\neg(\forall x)P(x) \to (\exists x)\neg P(x)) \\ | \\ T(\neg(\forall x)P(x)) \\ | \\ F(\exists x)\neg P(x) \end{array}} \\
| \\
\boxed{\begin{array}{c} T(\neg(\forall x)P(x)) \\ | \\ F(\forall x)P(x) \end{array}} \\
| \\
\boxed{\begin{array}{c} F(\forall x)P(x) \\ | \\ FP(d) \quad d \text{ new} \end{array}} \\
| \\
\boxed{\begin{array}{c} F(\exists x)\neg P(x) \\ | \quad \text{choose} \\ F(\neg P(d)) \quad t = d \end{array}} \\
| \\
\boxed{\begin{array}{c} F(\neg P(d)) \\ | \\ TP(d) \end{array}} \\
| \\
\otimes
\end{array}
$$

Convention

$$
\begin{array}{c}
F((\exists x)\neg P(x) \to \neg(\forall x)P(x)) \\
| \\
T(\exists x)\neg P(x) \\
| \\
F(\neg(\forall x)P(x)) \\
| \\
T(\forall x)P(x) \\
| \\
T(\neg P(c))\ c\ \ \text{nové} \\
| \\
FP(c) \\
| \\
\boxed{\begin{array}{c} T(\forall x)P(x) \\ | \\ TP(c) \end{array}} \\
| \\
\otimes
\end{array}
\qquad
\begin{array}{c}
F(\neg(\forall x)P(x) \to (\exists x)\neg P(x)) \\
| \\
T(\neg(\forall x)P(x)) \\
| \\
F(\exists x)\neg P(x) \\
| \\
F(\forall x)P(x) \\
| \\
FP(d) \quad d\ \ \text{nové} \\
| \\
\boxed{\begin{array}{c} F(\exists x)\neg P(x) \\ | \\ F(\neg P(d)) \end{array}} \\
| \\
TP(d) \\
| \\
\otimes
\end{array}
$$

We will not write the entry that is expanded again on the branch, except in cases when the entry is in the form of $T(\forall x)\varphi(x)$ or $F(\exists x)\varphi(x)$.

Tableau proofs

- A branch V in a tableau τ is *contradictory* if it contains entries $T\varphi$ and $F\varphi$ for some sentence φ, otherwise V is *noncontradictory*.
- A tableau τ is *contradictory* if every branch in τ is contradictory.
- A *tableau proof* (*proof by tableau*) of a sentence φ from a theory T is a contradictory tableau from T with $F\varphi$ in the root.
- A sentence φ is *(tableau) provable* from T, denoted by $T \vdash \varphi$, if it has a tableau proof from T.
- A *refutation* of a sentence φ by *tableau* from a theory T is a contradictory tableau from T with the root entry $T\varphi$.
- A sentence φ is *(tableau) refutable* from T if it has a refutation by tableau from T, i.e. $T \vdash \neg\varphi$.

Examples

- $F((\forall x)(P(x) \to Q(x)) \to ((\forall x)P(x) \to (\forall x)Q(x)))$
- $T(\forall x)(P(x) \to Q(x))$
- $F((\forall x)P(x) \to (\forall x)Q(x))$
- $T(\forall x)P(x)$
- $F(\forall x)Q(x)$
- $FQ(c)$ $\quad c$ nová
- $T(\forall x)P(x)$
- $TP(c)$
- $T(\forall x)(P(x) \to Q(x))$
- $T(P(c) \to Q(c))$
  - $FP(c)$
    - $\otimes$
  - $TQ(c)$
    - $\otimes$

- $F((\forall x)(\varphi(x) \wedge \psi(x)) \leftrightarrow ((\forall x)\varphi(x) \wedge (\forall x)\psi(x)))$
  - $T((\forall x)(\varphi(x) \wedge \psi(x)))$
  - $F((\forall x)\varphi(x) \wedge (\forall x)\psi(x))$
    - $F(\forall x)\varphi(x)$
    - $F\varphi(c)$ $\quad c$ nová
    - $T((\forall x)(\varphi(x) \wedge \psi(x)))$
    - $T(\varphi(c) \wedge \psi(c))$
    - $T\varphi(c)$
    - $T\psi(c)$
    - $\otimes$

    - $F(\forall x)\psi(x)$
    - $F\psi(d)$ $\quad d$ nová
    - $T((\forall x)(\varphi(x) \wedge \psi(x)))$
    - $T(\varphi(d) \wedge \psi(d))$
    - $T\varphi(d)$
    - $T\psi(d)$
    - $\otimes$

  - $F((\forall x)(\varphi(x) \wedge \psi(x)))$
  - $T((\forall x)\varphi(x) \wedge (\forall x)\psi(x))$
  - $T(\forall x)\varphi(x)$
  - $T(\forall x)\psi(x)$
  - $F(\varphi(e) \wedge \psi(e))$ $\quad e$ nová
    - $F\varphi(e)$
    - $T(\forall x)\varphi(x)$
    - $T\varphi(e)$
    - $\otimes$

    - $F\psi(e)$
    - $T(\forall x)\psi(x)$
    - $T\psi(e)$
    - $\otimes$

Finished tableaux

A finished noncontradictory branch should provide us with a counterexample.

An occurrence of an entry P in a node v of a tableau τ is *i-th* if v has exactly $i-1$ predecessors labeled by P; and is *reduced* on a branch V through v if

a) P is neither in form of $T(\forall x)\varphi(x)$ nor $F(\exists x)\varphi(x)$ and P occurs on V as a root of an atomic tableau, i.e. it was already expanded on V, or

b) P is in form of $T(\forall x)\varphi(x)$ or $F(\exists x)\varphi(x)$, P has an $(i+1)$-th occurrence on V, and V contains an entry $T\varphi(x/t_i)$ resp. $F\varphi(x/t_i)$ where t_i is the i-th ground term (of the language L_C).

Let V be a branch in a tableau τ from a theory T. We say that

- V is *finished* if it is contradictory, or every occurrence of an entry on V is reduced on V and, moreover, V contains $T\varphi$ for every $\varphi \in T$,
- τ is *finished* if every branch in τ is finished.

Systematic tableaux - construction

Let R be an entry and $T = \{\varphi_0, \varphi_1, \dots\}$ be a (possibly infinite) theory.

(1) We take the atomic tableau for R as τ_0. In case $(*)$ we choose any $c \in L_C \setminus L$, in case $(\sharp)$ we take t_1 for t. Till possible, proceed as follows.

(2) Let v be the leftmost node in the smallest level as possible in tableau τ_n containing an occurrence of an entry P that is not reduced on some noncontradictory branch through v. (If v does not exist, we take $\tau'_n = \tau_n$.)

(3a) If P is neither $T(\forall x)\varphi(x)$ nor $F(\exists x)\varphi(x)$, let τ'_n be the tableau obtained from τ_n by adjoining the atomic tableau for P to every noncontradictory branch through v. In case $(*)$ we choose c_i for the smallest possible i.

(3b) If P is $T(\forall x)\varphi(x)$ or $F(\exists x)\varphi(x)$ and it has i-th occurrence in v, let τ'_n be the tableau obtained from τ_n by adjoining atomic tableau for P to every noncontradictory branch through v, where we take the term t_i for t.

(4) Let τ_{n+1} be the tableau obtained from τ'_n by adjoining $T\varphi_n$ to every noncontradictory branch that does not contain $T\varphi_n$ yet. (If φ_n does not exist, we take $\tau_{n+1} = \tau'_n$.)

The *systematic tableau* for R from T is the result $\tau = \cup\tau_n$ of this construction.