# Propositional and Predicate Logic - X

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# Extensions of theories

*We show that introducing new definitions has only an "auxiliary character".*

**Proposition** *Let $T$ be a theory of $L$ and $T'$ be a theory of $L'$ where $L \subseteq L'$.*

$(i)$ *$T'$ is an extension of $T$ if and only if the reduct $\mathcal{A}$ of every model $\mathcal{A}'$ of $T'$ to the language $L$ is a model of $T$,*

$(ii)$ *$T'$ is a conservative extension of $T$ if $T'$ is an extension of $T$ and every model $\mathcal{A}$ of $T$ can be expanded to the language $L'$ on a model $\mathcal{A}'$ of $T'$.*

*Proof*

$(i)a)$ If $T'$ is an extension of $T$ and $\varphi$ is any axiom of $T$, then $T' \models \varphi$. Thus $\mathcal{A}' \models \varphi$ and also $\mathcal{A} \models \varphi$, which implies that $\mathcal{A}$ is a model of $T$.

$(i)b)$ If $\mathcal{A}$ is a model of $T$ and $T \models \varphi$ where $\varphi$ is of $L$, then $\mathcal{A} \models \varphi$ and also $\mathcal{A}' \models \varphi$. This implies that $T' \models \varphi$ and thus $T'$ is an extension of $T$.

$(ii)$ If $T' \models \varphi$ where $\varphi$ is of $L$ and $\mathcal{A}$ is a model of $T$, then in its expansion $\mathcal{A}'$ that models $T'$ we have $\mathcal{A}' \models \varphi$. Thus also $\mathcal{A} \models \varphi$, and hence $T \models \varphi$. Therefore $T'$ is conservative. $\square$

# Extensions by definition of a relation symbol

Let $T$ be a theory of $L$, $\psi(x_1, \dots, x_n)$ be a formula of $L$ in free variables $x_1, \dots, x_n$ and $L'$ denote the language $L$ with a new $n$-ary relation symbol $R$.

The *extension* of $T$ *by definition of $R$* with the formula $\psi$ is the theory $T'$ of $L'$ obtained from $T$ by adding the axiom

$$R(x_1, \dots, x_n) \;\leftrightarrow\; \psi(x_1, \dots, x_n)$$

**Observation** *Every model of $T$ can be uniquely expanded to a model of $T'$.*

**Corollary** *$T'$ is a conservative extension of $T$.*

**Proposition** *For every formula $\varphi'$ of $L'$ there is $\varphi$ of $L$ s.t. $T' \models \varphi' \leftrightarrow \varphi$.*

*Proof* Replace each subformula $R(t_1, \dots, t_n)$ in $\varphi$ with $\psi'(x_1/t_1, \dots, x_n/t_n)$, where $\psi'$ is a suitable variant of $\psi$ allowing all substitutions. $\square$

*For example, the symbol $\leq$ can be defined in arithmetics by the axiom*

$$x \leq y \;\leftrightarrow\; (\exists z)(x + z = y)$$

# Extensions by definition of a function symbol

Let $T$ be a theory of a language $L$ and $\psi(x_1, \ldots, x_n, y)$ be a formula of $L$ in free variables $x_1, \ldots, x_n, y$ such that

$$T \models (\exists y)\psi(x_1, \ldots, x_n, y) \qquad \text{(existence)}$$

$$T \models \psi(x_1, \ldots, x_n, y) \ \wedge\ \psi(x_1, \ldots, x_n, z) \ \rightarrow\ y = z \qquad \text{(uniqueness)}$$

Let $L'$ denote the language $L$ with a new $n$-ary function symbol $f$.

The *extension* of $T$ *by definition of $f$* with the formula $\psi$ is the theory $T'$ of $L'$ obtained from $T$ by adding the axiom

$$f(x_1, \ldots, x_n) = y \ \leftrightarrow\ \psi(x_1, \ldots, x_n, y)$$

*Remark* *In particular, if $\psi$ is $t(x_1, \ldots, x_n) = y$ where $t$ is a term and $x_1, \ldots, x_n$ are the variables in $t$, both the conditions of existence and uniqueness hold.*

*For example binary $-$ can be defined using $+$ and unary $-$ by the axiom*

$$x - y = z \ \leftrightarrow\ x + (-y) = z$$

# Extensions by definition of a function symbol (cont.)

**Observation** *Every model of $T$ can be uniquely expanded to a model of $T'$.*

**Corollary** *$T'$ is a conservative extension of $T$.*

**Proposition** *For every formula $\varphi'$ of $L'$ there is $\varphi$ of $L$ s.t. $T' \models \varphi' \leftrightarrow \varphi$.*

*Proof* It suffices to consider $\varphi'$ with a single occurrence of $f$. If $\varphi'$ has more, we may proceed inductively. Let $\varphi^*$ denote the formula obtained from $\varphi'$ by replacing the term $f(t_1, \ldots, t_n)$ with a new variable $z$. Let $\varphi$ be the formula

$$(\exists z)(\varphi^* \ \wedge \ \psi'(x_1/t_1, \ldots, x_n/t_n, y/z)),$$

where $\psi'$ is a suitable variant of $\psi$ allowing all substitutions.

Let $\mathcal{A}$ be a model of $T'$, $e$ be an assignment, and $a = f^A(t_1, \ldots, t_n)[e]$. By the two conditions, $\mathcal{A} \models \psi'(x_1/t_1, \ldots, x_n/t_n, y/z)[e]$ if and only if $e(z) = a$. Thus

$$\mathcal{A} \models \varphi[e] \quad \Leftrightarrow \quad \mathcal{A} \models \varphi^*[e(z/a)] \quad \Leftrightarrow \quad \mathcal{A} \models \varphi'[e]$$

for every assignment $e$, i.e. $\mathcal{A} \models \varphi' \leftrightarrow \varphi$ and so $T' \models \varphi' \leftrightarrow \varphi$. $\square$

# Extensions by definitions

A theory $T'$ of $L'$ is called an *extension* of a theory $T$ of $L$ *by definitions* if it is obtained from $T$ by successive definitions of relation and function symbols.

**Corollary** *Let $T'$ be an extension of a theory $T$ by definitions. Then*

- *every model of $T$ can be uniquely expanded to a model of $T'$,*
- *$T'$ is a conservative extension of $T$,*
- *for every formula $\varphi'$ of $L'$ there is a formula $\varphi$ of $L$ such that $T' \models \varphi' \leftrightarrow \varphi$.*

*For example, in $T = \{(\exists y)(x + y = 0), (x + y = 0) \wedge (x + z = 0) \rightarrow y = z\}$ of $L = \langle +, 0, \leq \rangle$ with equality we can define $<$ and unary $-$ by the axioms*

$$-x = y \quad \leftrightarrow \quad x + y = 0$$
$$x < y \quad \leftrightarrow \quad x \leq y \ \wedge \ \neg(x = y)$$

*Then the formula $-x < y$ is equivalent in this extension to a formula*

$$(\exists z)((z \leq y \ \wedge \ \neg(z = y)) \ \wedge \ x + z = 0).$$

# Equisatisfiability

*We will see that the problem of satisfiability can be reduced to open theories.*

- Theories $T$, $T'$ are *equisatisfiable* if $T$ has a model $\Leftrightarrow$ $T'$ has a model.
- A formula $\varphi$ is in the *prenex (normal) form (PNF)* if it is written as

  $$(Q_1 x_1) \dots (Q_n x_n)\varphi',$$

  where $Q_i$ denotes $\forall$ or $\exists$, variables $x_1, \dots, x_n$ are all distinct and $\varphi'$ is an open formula, called the *matrix*. $(Q_1 x_1) \dots (Q_n x_n)$ is called the *prefix*.
- In particular, if all quantifiers are $\forall$, then $\varphi$ is a *universal* formula.

*To find an open theory equisatisfiable with $T$ we proceed as follows.*

(1) We replace axioms of $T$ by equivalent formulas in the prenex form.

(2) We transform them, using new function symbols, to equisatisfiable universal formulas, so called Skolem variants.

(3) We take their matrices as axioms of a new theory.

# Conversion rules for quantifiers

Let $Q$ denote $\forall$ or $\exists$ and let $\overline{Q}$ denote the complementary quantifier.
For every formulas $\varphi$, $\psi$ such that $x$ in not free in the formula $\psi$,

$$\begin{array}{lrcl}
\models & \neg(Qx)\varphi & \leftrightarrow & (\overline{Q}x)\neg\varphi \\
\models & ((Qx)\varphi \wedge \psi) & \leftrightarrow & (Qx)(\varphi \wedge \psi) \\
\models & ((Qx)\varphi \vee \psi) & \leftrightarrow & (Qx)(\varphi \vee \psi) \\
\models & ((Qx)\varphi \rightarrow \psi) & \leftrightarrow & (\overline{Q}x)(\varphi \rightarrow \psi) \\
\models & (\psi \rightarrow (Qx)\varphi) & \leftrightarrow & (Qx)(\psi \rightarrow \varphi)
\end{array}$$

The above equivalences can be verified semantically or proved by the tableau method (*by taking the universal closure if it is not a sentence*).

*Remark* *The assumption that $x$ is not free in $\psi$ is necessary in each rule above (except the first one) for some quantifier $Q$. For example,*

$$\not\models \ ((\exists x)P(x) \wedge P(x)) \ \leftrightarrow \ (\exists x)(P(x) \wedge P(x))$$

# Conversion to the prenex normal form

**Proposition** *Let $\varphi'$ be the formula obtained from $\varphi$ by replacing some occurrences of a subformula $\psi$ with $\psi'$. If $T \models \psi \leftrightarrow \psi'$, then $T \models \varphi \leftrightarrow \varphi'$.*

*Proof* Easily by induction on the structure of the formula $\varphi$. □

**Proposition** *For every formula $\varphi$ there is an equivalent formula $\varphi'$ in the prenex normal form, i.e. $\models \varphi \leftrightarrow \varphi'$.*

*Proof* By induction on the structure of $\varphi$ applying the conversion rules for quantifiers, replacing subformulas with their variants if needed, and applying the above proposition on equivalent transformations. □

*For example,*

$$\begin{aligned}
((\forall z)P(x,z) \wedge P(y,z)) &\rightarrow \neg(\exists x)P(x,y) \\
((\forall u)P(x,u) \wedge P(y,z)) &\rightarrow (\forall x)\neg P(x,y) \\
(\forall u)(P(x,u) \wedge P(y,z)) &\rightarrow (\forall v)\neg P(v,y) \\
(\exists u)((P(x,u) \wedge P(y,z)) &\rightarrow (\forall v)\neg P(v,y)) \\
(\exists u)(\forall v)((P(x,u) \wedge P(y,z)) &\rightarrow \neg P(v,y))
\end{aligned}$$

# Skolem variants

Let $\varphi$ be a sentence of a language $L$ in the prenex normal form, let $y_1, \ldots, y_n$ be the existentially quantified variables in $\varphi$ (in this order), and for every $i \leq n$ let $x_1, \ldots, x_{n_i}$ be the variables that are universally quantified in $\varphi$ before $y_i$. Let $L'$ be an extension of $L$ with new $n_i$-ary function symbols $f_i$ for all $i \leq n$.

Let $\varphi_S$ denote the formula of $L'$ obtained from $\varphi$ by removing all $(\exists y_i)$'s from the prefix and by replacing each occurrence of $y_i$ with the term $f_i(x_1, \ldots, x_{n_i})$. Then $\varphi_S$ is called a *Skolem variant* of $\varphi$.

*For example, for the formula* $\varphi$

$$(\exists y_1)(\forall x_1)(\forall x_2)(\exists y_2)(\forall x_3)R(y_1, x_1, x_2, y_2, x_3)$$

*the following formula* $\varphi_S$ *is a Skolem variant of* $\varphi$

$$(\forall x_1)(\forall x_2)(\forall x_3)R(f_1, x_1, x_2, f_2(x_1, x_2), x_3),$$

*where* $f_1$ *is a new constant symbol and* $f_2$ *is a new binary function symbol.*

# Properties of Skolem variants

**Lemma** *Let $\varphi$ be a sentence $(\forall x_1)\dots(\forall x_n)(\exists y)\psi$ of $L$ and $\varphi'$ be a sentence $(\forall x_1)\dots(\forall x_n)\psi(y/f(x_1,\dots,x_n))$ where $f$ is a new function symbol. Then*

(1) *the reduct $\mathcal{A}$ of every model $\mathcal{A}'$ of $\varphi'$ to the language $L$ is a model of $\varphi$,*

(2) *every model $\mathcal{A}$ of $\varphi$ can be expanded into a model $\mathcal{A}'$ of $\varphi'$.*

*Remark* *Compared to extensions by definition of a function symbol, the expansion in (2) does not need to be unique now.*

*Proof* (1) Let $\mathcal{A}' \models \varphi'$ and $\mathcal{A}$ be the reduct of $\mathcal{A}'$ to $L$. Since $\mathcal{A} \models \psi[e(y/a)]$ for every assignment $e$ where $a = (f(x_1,\dots,x_n))^{A'}[e]$, we have also $\mathcal{A} \models \varphi$.
(2) Let $\mathcal{A} \models \varphi$. There exists a function $f^A\colon A^n \to A$ such that for every assignment $e$ it holds $\mathcal{A} \models \psi[e(y/a)]$ where $a = f^A(e(x_1),\dots,e(x_n))$, and thus the expansion $\mathcal{A}'$ of $\mathcal{A}$ by the function $f^A$ is a model of $\varphi'$. $\square$

**Corollary** *If $\varphi'$ is a Skolem variant of $\varphi$, then both statements (1) and (2) hold for $\varphi$, $\varphi'$ as well. Hence $\varphi$, $\varphi'$ are equisatisfiable.*

# Skolem's theorem

**Theorem** *Every theory $T$ has an open conservative extension $T^*$.*

*Proof* We may assume that $T$ is in a closed form. Let $L$ be its language.

- By replacing each axiom of $T$ with an equivalent formula in the prenex normal form we obtain an equivalent theory $T^\circ$.
- By replacing each axiom of $T^\circ$ with its Skolem variant we obtain a theory $T'$ in an extended language $L' \supseteq L$.
- Since the reduct of every model of $T'$ to the language $L$ is a model of $T$, the theory $T'$ is an extension of $T$.
- Furthermore, since every model of $T$ can be expanded to a model of $T'$, it is a conservative extension.
- Since every axiom of $T'$ is a universal sentence, by replacing them with their matrices we obtain an open theory $T^*$ equivalent to $T'$. $\square$

**Corollary** *For every theory there is an equisatisfiable open theory.*

# Reduction of unsatisfiability to propositional logic

*If an open theory is unsatisfiable, we can demonstrate it "via ground terms".*

For example, in the language $L = \langle P, R, f, c\rangle$ the theory

$$T = \{P(x,y) \vee R(x,y),\ \neg P(c,y),\ \neg R(x,f(x))\}$$

is unsatisfiable, and this can be demonstrated by an unsatisfiable conjunction of finitely many instances of (some) axioms of $T$ in ground terms

$$(P(c,f(c)) \vee R(c,f(c)))\ \wedge\ \neg P(c,f(c))\ \wedge\ \neg R(c,f(c)),$$

which may be seen as an unsatisfiable propositional formula

$$(p \vee r)\ \wedge\ \neg p\ \wedge\ \neg r.$$

An instance $\varphi(x_1/t_1, \ldots, x_n/t_n)$ of an open formula $\varphi$ in free variables $x_1, \ldots, x_n$ is a *ground instance* if all terms $t_1, \ldots, t_n$ are ground terms (i.e. terms without variables).

# Herbrand model

Let $L = \langle \mathcal{R}, \mathcal{F} \rangle$ be a language with at least one constant symbol. *(If needed, we add a new constant symbol to $L$.)*

- The *Herbrand universe* for $L$ is the set of all ground terms of $L$.
  *For example, for $L = \langle P, f, c \rangle$ with $f$ binary function sym., $c$ constant sym.*
  $$A = \{c, f(c,c), f(f(c,c),c), f(c, f(c,c)), f(f(c,c), f(c,c)), \dots\}$$
- An $L$-structure $\mathcal{A}$ is a *Herbrand structure* if its domain $A$ is the Herbrand universe for $L$ and for each $n$-ary function symbol $f \in \mathcal{F}$, $t_1, \dots, t_n \in A$,
  $$f^A(t_1, \dots, t_n) = f(t_1, \dots, t_n)$$
  (including $n = 0$, i.e. $c^A = c$ for every constant symbol $c$).
  *Remark* *Compared to a canonical model, the relations are not specified.*
  *E.g.* $\mathcal{A} = \langle A, P^A, f^A, c^A \rangle$ *with* $P^A = \emptyset$, $c^A = c$, $f^A(c,c) = f(c,c)$, $\dots$.
- A *Herbrand model* of a theory $T$ is a Herbrand structure that models $T$.

# Herbrand's theorem

**Theorem** *Let $T$ be an open theory of a language $L$ without equality and with at least one constant symbol. Then*

$(a)$ *either $T$ has a Herbrand model, or*

$(b)$ *there are finitely many ground instances of axioms of $T$ whose conjunction is unsatisfiable, and thus $T$ has no model.*

*Proof* Let $T'$ be the set of all ground instances of axioms of $T$. Consider a finished (e.g. systematic) tableau $\tau$ from $T'$ in the language $L$ (without adding new constant symbols) with the root entry $F\bot$.

- If the tableau $\tau$ contains a noncontradictory branch $V$, the canonical model from $V$ is a Herbrand model of $T$.
- Else, $\tau$ is contradictory, i.e. $T' \vdash \bot$. Moreover, $\tau$ is finite, so $\bot$ is provable from finitely many formulas of $T'$, i.e. their conjunction is unsatisfiable. $\square$

*Remark* *If the language $L$ is with equality, we extend $T$ to $T^*$ by axioms of equality for $L$ and if $T^*$ has a Herbrand model $\mathcal{A}$, we take its quotient by $=^A$.*

# Corollaries of Herbrand's theorem

Let $L$ be a language containing at least one constant symbol.

**Corollary** *For every open $\varphi(x_1, \ldots, x_n)$ of $L$, the formula $(\exists x_1) \ldots (\exists x_n)\varphi$ is valid if and only if there exist $mn$ ground terms $t_{ij}$ of $L$ for some $m$ such that*

$$\varphi(x_1/t_{11}, \ldots, x_n/t_{1n}) \vee \ldots \vee \varphi(x_1/t_{m1}, \ldots, x_n/t_{mn})$$

*is a (propositional) tautology.*

*Proof* $(\exists x_1) \ldots (\exists x_n)\varphi$ is valid $\Leftrightarrow$ $(\forall x_1) \ldots (\forall x_n)\neg\varphi$ is unsatisfiable $\Leftrightarrow$ $\neg\varphi$ is unsatisfiable. The rest follows from Herbrand's theorem for $\{\neg\varphi\}$. $\square$

**Corollary** *An open theory $T$ of $L$ is satisfiable if and only if the theory $T'$ of all ground instances of axioms of $T$ is satisfiable.*

*Proof* If $T$ has a model $\mathcal{A}$, every instance of each axiom of $T$ is valid in $\mathcal{A}$, thus $\mathcal{A}$ is a model of $T'$. If $T$ is unsatisfiable, by H. theorem there are (finitely) formulas of $T'$ whose conjunction is unsatisfiable, thus $T'$ is unsatisfiable. $\square$

# Resolution method in predicate logic - introduction

- A refutation procedure - its aim is to show that a given formula (or theory) is unsatisfiable.
- It assumes open formulas in CNF (and in clausal form).

  A *literal* is *(now)* an atomic formula or its negation.

  A *clause* is a finite set of literals, $\square$ denotes the empty clause.

  A *formula (in clausal form)* is a (possibly infinite) set of clauses.

  *Remark* *Every formula (theory) can be converted to an equisatisfiable open formula (theory) in CNF, and then to a formula in clausal form.*
- The resolution rule is more general - it allows to resolve through literals that are unifiable.
- Resolution in predicate logic is based on resolution in propositional logic and unification.

# Reduction to propositional level (grounding)

*Herbrand's theorem gives us the following (inefficient) method.*

- Let $S$ be the *(input)* formula in clausal form.
- We can assume that the language contains at least one constant symbol.
- Let $S'$ be the set of all ground instances of all clauses from $S$.
- By introducing propositional letters representing atomic sentences we may view $S'$ as a (possibly infinite) propositional formula in clausal form.
- We may verify that it is unsatisfiable by resolution on propositional level.

*For example, for* $S = \{\{P(x,y), R(x,y)\}, \{\neg P(c,y)\}, \{\neg R(x,f(x))\}\}$ *the set*

$$S' = \{\{P(c,c), R(c,c)\}, \{P(c,f(c)), R(c,f(c))\}, \{P(f(c),f(c)), R(f(c),f(c))\} \ldots, \{\neg P(c,c)\}, \{\neg P(c,f(c))\}, \ldots, \{\neg R(c,f(c))\}, \{\neg R(f(c),f(f(c)))\}, \ldots\}$$

*is unsatisfiable since on propositional level*

$$S' \supseteq \{\{P(c,f(c)), R(c,f(c))\}, \{\neg P(c,f(c))\}, \{\neg R(c,f(c))\}\} \vdash_R \square.$$

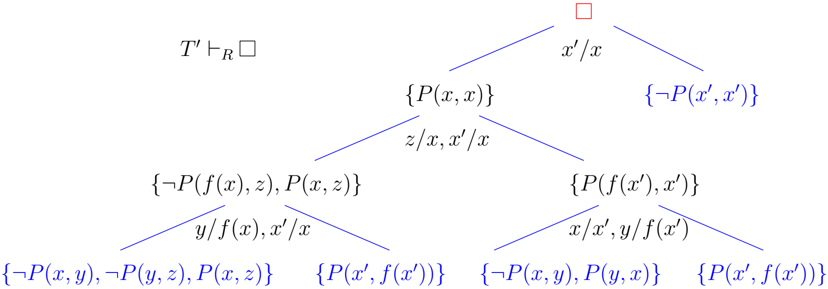
# Resolution in predicate logic - an example

*But we do not know which ground instances to use. Instead, we proceed on a higher level applying substitutions that unify literals to be resolved.*

Consider $T = \{\neg P(x,x),\ P(x,y) \rightarrow P(y,x),\ P(x,y) \wedge P(y,z) \rightarrow P(x,z)\}$.

Is $T \models (\exists x)\neg P(x, f(x))$ ? Equivalently, is the following $T'$ unsatisfiable?

$T' = \{\{\neg P(x,x)\}, \{\neg P(x,y), P(y,x)\}, \{\neg P(x,y), \neg P(y,z), P(x,z)\}, \{P(x, f(x))\}\}$



# Hilbert's calculus in predicate logic

- basic connectives and quantifier: $\neg$, $\rightarrow$, $(\forall x)$ (others are derived)
- allows to prove any formula (not just sentences)
- *logical axioms* (schemes of axioms):

$$
\begin{array}{rll}
(i) & \varphi \rightarrow (\psi \rightarrow \varphi) & \\
(ii) & (\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi)) & \\
(iii) & (\neg\varphi \rightarrow \neg\psi) \rightarrow (\psi \rightarrow \varphi) & \\
(iv) & (\forall x)\varphi \rightarrow \varphi(x/t) & \text{if } t \text{ is substitutable for } x \text{ to } \varphi \\
(v) & (\forall x)(\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow (\forall x)\psi) & \text{if } x \text{ is not free in } \varphi
\end{array}
$$

  where $\varphi$, $\psi$, $\chi$ are any formulas (of a given language), $t$ is any term, and $x$ is any variable
- in a language with equality we include also the axioms of equality
- *rules of inference*

$$\frac{\varphi,\ \varphi \rightarrow \psi}{\psi} \quad \text{(modus ponens)}, \qquad \frac{\varphi}{(\forall x)\varphi} \quad \text{(generalization)}$$

# Hilbert-style proofs

A *proof* (in *Hilbert-style*) of a formula $\varphi$ from a theory $T$ is a finite sequence $\varphi_0, \ldots, \varphi_n = \varphi$ of formulas such that for every $i \leq n$

- $\varphi_i$ is a logical axiom or $\varphi_i \in T$ (an axiom of the theory), or
- $\varphi_i$ can be inferred from the previous formulas applying a rule of inference.

A formula $\varphi$ is *provable* from $T$ if it has a proof from $T$, denoted by $T \vdash_H \varphi$.

**Theorem** (soundness) *For every theory $T$ and formula $\varphi$, $T \vdash_H \varphi \;\Rightarrow\; T \models \varphi$.*

*Proof*

- If $\varphi$ is an axiom (logical or from $T$), then $T \models \varphi$ (l. axioms are tautologies),
- if $T \models \varphi$ and $T \models \varphi \rightarrow \psi$, then $T \models \psi$, i.e. modus ponens is sound,
- if $T \models \varphi$, then $T \models (\forall x)\varphi$, i.e. generalization is sound,
- thus every formula in a proof from $T$ is valid in $T$. $\square$

*Remark* *The completeness holds as well, i.e. $T \models \varphi \Rightarrow T \vdash_H \varphi$.*