Propositional and Predicate Logic - Exam Test

January 10, 2014

1. *(Pigeonhole principle).* Let $n \geq 2$ be a fixed integer. Assume that we have n pigeons and $n-1$ holes. We want to prove (by resolution) that it is impossible (at the same time) that

   (i) every pigeon sits in some hole,

   (ii) every hole hosts at most one pigeon.

   Let $\mathbb{P} = \{p^i_j \mid 1 \leq i \leq n, 1 \leq j \leq n-1\}$ be the set of propositional letters where p^i_j represents that *"the i-th pigeon sits in the j-th hole"*.

   (a) Write propositions φ_i and ψ_j over $\mathbb{P}$ expressing that *"the i-th pigeon sits in some hole"*, resp. *"the j-th hole hosts at most one pigeon"* where $1 \leq i \leq n$, $1 \leq j \leq n-1$. Find a theory T_n based on propositions φ_i and φ_j expressing (i) and (ii). *(2pts)*

   (b) Now, let $n = 3$ and $T' = T_3 \cup \{p^1_1\}$, i.e. we additionally assume that *"the first pigeon sits in the first hole"*. Transform T' into the set representation (i.e. clausal form). *(2pts)*

   (c) Show that $T' \vdash_R \square$. Write down your resolution refutation as a resolution tree. Underline the resolved literals at each step. *(4pts)*

   (d) Let $T^* = T' \setminus \{\psi_2\}$ be a theory over $\mathbb{P}$. Is T' a conservative extension of T^*? Give an explanation. *(2pts)*

2. Let $T = \{(\forall x)(\exists y)(\forall z)(\neg P(x) \vee Q(y,z)), (\exists y)(\forall x)\neg Q(x,y), (\exists x)P(x)\}$ be a theory of a language $L = \langle P, Q\rangle$ without equality where P, Q is unary resp. binary relation symbol.

   (a) Applying skolemization find a theory T' (in a suitably extended language) that is equisatisfiable with T and that has only universal sentences as axioms. *(2pts)*

   (b) Prove by tableau method that T' is unsatisfiable. *(4pts)*

   (c) Let T'' be the set of matrices of axioms of T', so T'' is an open theory. Find a conjunction of ground instances of axioms of T'' that is unsatisfiable. *Hint: use the tableau from (b). (2pts)*

   (d) Is T complete? What are its (mutually nonequivalent) simple complete extensions? Give an explanation. *(2pts)*

3. Let $T = \{\varphi\}$ be a theory of a language $L = \langle U\rangle$ with equality where U is a unary relation symbol and the axiom φ expresses that *"$U(x)$ holds exactly for 42 elements."*

   (a) Are all infinitely countable models of T elementarily equivalent? Give an explanation. *(2pts)*

   (b) Find two nonequivalent simple complete extensions of T or show that such extensions do not exist. *(2pts)*

   (c) Is T equivalent to an open theory? Give an explanation. *(2pts)*

   (d) Let $T' = \{\psi\}$ be a theory of a language $L' = \langle\rangle$ with equality where ψ expresses that *"There exists at least 42 elements."* Is T a conservative extension of T'? Give an explanation. *(2pts)*