Predicate and Propositional Logic - Seminar 1

Oct 6, 2014

1. Recall basic definitions from the first lecture with more examples whenever needed.
   (a) Show that for every tree T (following the definition from the lecture) for every two nodes x, y with $x <_T y$ there exists an *immediate successor* (a son) of x between x and y.
   (b) Give an example of a tree in which some node (except the root) has no immediate predecessor (a father).
   (c) Show that every finitely branching tree in which every node (except the root) has a father is at most countable.
2. Consider a finite game of two alternating players. Assume the game ends after n rounds by win of one of the players, denoted by X, Y where X begins. The game is given by formula $\varphi(x_1, y_1, x_2, y_2, \ldots, x_n, y_n)$ expressing that the game with moves $x_1, y_1, x_2, y_2, \ldots, x_n, y_n$ ends by a win of X. Find a formula (with use of quantifiers) that expresses
   (a) "X cannot lose", "Y cannot lose",
   (b) "X has a winning strategy",
   (c) "Y has a winning strategy".
3. Assume we are given an (undirected) graph G and two his vertices u, v. Find a propositional formula that is satisfiable if and only if
   (a) G is bipartite,
   (b) G has a perfect matching,
   (c) there is a path between u and v in G.
4. Find first-order formulae (in the language of graph theory) expressing about a graph that
   (a) "u and v have at most one common neighbor",
   (b) "there are at least three mutually independent edges",
   (c) "there is a path of length n between u and v, where given $n > 0$ is fixed.
5. Find second-order formulae (in the language of graph theory) expressing about a graph that
   (a) "there exists a bipartition",
   (b) "there exists a perfect matching",
   (c) "there exists a path between u and v".
6. Find first-order formulae (with the symbol $\leq$) expressing about a partially ordered set
   (a) "x is the smallest element", "x is a minimal element",
   (b) "x has an immediate successor",
   (c) "every two elements have the least common predecessor".
7. Find first order formulae (with use of equality) expressing for a fixed $n > 0$ that
   (a) "there exist at least n elements",
   (b) "there exist at most n elements",
   (c) "there exist exactly n elements"

   Is it possible to express with use of (possibly infinite) set of formulae that "there are infinitely many elements"?

8. Find a second-order formula expressing “there exist finitely many elements”. Hint:

   (a) Find first-order formulae (with a symbol f for a function) expressing “f is injective”, “f is surjective”.

   (b) Find a second-order formula expressing “every function that is surjective is also injective”.