Predicate and Propositional Logic - Seminar 2

Oct 13, 2014

1. Find first-order formulae (with the symbol $\leq$) expressing about a partially ordered set
   (a) "x is the smallest element", "x is a minimal element",
   (b) "x has an immediate successor",
   (c) "every two elements have the least common predecessor".
2. Find first order formulae (with use of equality) expressing for a fixed $n > 0$ that
   (a) "there exist at least n elements",
   (b) "there exist at most n elements",
   (c) "there exist exactly n elements"

   Is it possible to express with use of (possibly infinite) set of formulae that "there are infinitely many elements"?
3. Find a second-order formula expressing "there exist finitely many elements". Hint:
   (a) Find first-order formulae (with a symbol f for a function) expressing "f is injective", "f is surjective".
   (b) Find a second-order formula expressing "every function that is surjective is also injective".
4. Prove or disprove that the following sets of connectives are adequate.
   (a) $\{\downarrow\}$ where $\downarrow$ is Peirce arrow (NOR)
   (b) $\{\uparrow\}$ where $\uparrow$ is Sheffer stroke (NAND)
   (c) $\{\vee, \rightarrow, \leftrightarrow\}$, $\{\vee, \wedge, \rightarrow\}$
5. Transform the following propositions into DNF and CNF a) by using truth tables (determining the models), b) by using transformation rules.
   (a) $(\neg p \vee q) \rightarrow (\neg q \wedge r)$
   (b) $(\neg p \rightarrow (\neg q \rightarrow r)) \rightarrow p$
   (c) $((p \rightarrow \neg q) \rightarrow \neg r) \rightarrow \neg p$
6. Applying the implication graph determine whether the following proposition in 2-CNF is satisfiable or not; and if yes, find a satisfying assignment.

$$(p_0 \vee p_2) \wedge (p_0 \vee \neg p_3) \wedge (p_1 \vee \neg p_3) \wedge (p_1 \vee \neg p_4) \wedge (p_2 \vee \neg p_4) \wedge (p_0 \vee \neg p_5) \wedge$$
$$(p_1 \vee \neg p_5) \wedge (p_2 \vee \neg p_5) \wedge (\neg p_1 \vee \neg p_6) \wedge (p_4 \vee p_6) \wedge (p_5 \vee p_6) \wedge p_1$$

7. Find both DNF and CNF representations of the Boolean function maj: ${}^{3}2 \rightarrow 2$ defined as the majority of the three (truth) values.
8. Let $\mathrm{maj}_n \colon {}^{3}({}^{n}2) \rightarrow {}^{n}2$ be the coordinate-wise majority function; that is, for example

$$\mathrm{maj}_4((0,1,0,1),(1,1,0,0),(1,1,0,0)) = (1,1,0,0)$$

   We say that a set $K \subseteq {}^{n}2$ is a *median* set if it is closed under maj_n.

   (a) Show that for every 2-CNF proposition φ it holds that $M(\varphi)$ is a median set.

   (b)* Show that for every median set $K \subseteq {}^{n}2$ there exists a 2-CNF proposition φ over n variables such that $M(\varphi) = K$.