Predicate and Propositional Logic - Seminar 8

Dec 1, 2014

1. Prove (semantically) the following claims. For every structure $\mathcal{A}$, formula φ, and sentence ψ,

   (a) $\mathcal{A} \models (\psi \rightarrow (\exists x)\varphi) \Leftrightarrow \mathcal{A} \models (\exists x)(\psi \rightarrow \varphi)$

   (b) $\mathcal{A} \models (\psi \rightarrow (\forall x)\varphi) \Leftrightarrow \mathcal{A} \models (\forall x)(\psi \rightarrow \varphi)$

   (c) $\mathcal{A} \models ((\exists x)\varphi \rightarrow \psi) \Leftrightarrow \mathcal{A} \models (\forall x)(\varphi \rightarrow \psi)$

   (d) $\mathcal{A} \models (\psi \rightarrow (\forall x)\varphi) \Leftrightarrow \mathcal{A} \models (\exists x)(\psi \rightarrow \varphi)$

   Does this hold also for every formula ψ with a free variable x? And for every formula ψ in which x is not free?

2. Determine whether the following holds for every formula φ.

   (a) $\varphi \models (\forall x)\varphi$

   (b) $\models \varphi \rightarrow (\forall x)\varphi$

   (c) $\varphi \models (\exists x)\varphi$

   (d) $\models \varphi \rightarrow (\exists x)\varphi$

3. The theory of groups T is of language $L = \langle +, -, 0\rangle$ with equality where $+$ is a binary function symbol, $-$ is a unary function symbol, 0 is a constant symbol, and has it axioms

$$\begin{gathered} x + (y + z) = (x + y) + z \\ 0 + x = x = x + 0 \\ x + (-x) = 0 = (-x) + x \end{gathered}$$

   Are the following formulas valid / contradictory / independent in T?

   (a) $x + y = y + x$

   (b) $x + y = x \;\rightarrow\; y = 0$

   (c) $x + y = 0 \;\rightarrow\; y = -x$

   (d) $-(x + y) = (-y) + (-x)$

4. Consider a structure $\underline{\mathbb{Z}_4} = \langle \{0, 1, 2, 3\}, +, -, 0\rangle$ where $+$ is the binary addition modulo 4 and $-$ is the unary function for the *inverse* element of $+$ with respect to the *neutral* element 0.

   (a) Is $\underline{\mathbb{Z}_4}$ a model of the theory T from the previous example (i.e. is it a *group*)?

   (b) Determine all substructures $\underline{\mathbb{Z}_4}\langle a\rangle$ generated by some $a \in \mathbb{Z}_4$.

   (c) Does $\underline{\mathbb{Z}_4}$ contain also other substructures?

   (d) Is every substructure of $\underline{\mathbb{Z}_4}$ a model of T?

   (e) Is every substructure of $\underline{\mathbb{Z}_4}$ elementary equivalent to $\underline{\mathbb{Z}_4}$?

   (f) Is every substructure of a *commutative* group (i.e. a group that satisfies also 9(a)) again a commutative group?

5. Let $\underline{\mathbb{Q}} = \langle \mathbb{Q}, +, -, \cdot, 0, 1\rangle$ be the structure of rational numbers with standard operations (thus forming a *field*).

   (a) Is there a reduct of $\underline{\mathbb{Q}}$ that is a model of T from the previous exercises?

   (b) Can we expand the reduct $\langle \mathbb{Q}, \cdot, 1\rangle$ to a model of T?

   (c) Does $\underline{\mathbb{Q}}$ contain a substructure that is not elementary equivalent to $\underline{\mathbb{Q}}$?

(d) Let $Th(\underline{\mathbb{Q}})$ denote the set of all sentences that are valid in $\underline{\mathbb{Q}}$. Is $Th(\underline{\mathbb{Q}})$ a complete theory?

6. Let $T = \{x = c_1 \vee x = c_2 \vee x = c_3\}$ be a theory of $L = \langle c_1, c_2, c_3 \rangle$ with equality.

   (a) Is T (semantically) consistent?

   (b) Are all models of T elementary equivalent? That is, is T (semantically) complete?

   (c) Find all simple complete extensions of T.

   (d) Is a theory $T' = T \cup \{x = c_1 \vee x = c_4\}$ of the language $L = \langle c_1, c_2, c_3, c_4 \rangle$ an extension of T? Is T' a simple extension of T? Is T' a conservative extension of T?